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ON A DIFFERENTIAL EQUATION OF BOUNDARY-LAYER THEORY

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CONTENTS

The differential equation $y''' + yy'' + \lambda(1-y'^2) = 0$ is of importance in boundary-layer theory. H. Weyl has shown that the equation has a solution which satisfies certain boundary conditions. The present paper gives a simple proof of Weyl's result and then investigates in detail the properties of all solutions of the equation.

1. The boundary-layer equation

According to the assumptions of boundary-layer theory (see Schlichting 1955) the steady two-dimensional flow of a slightly viscous incompressible fluid past a wedge of angle $\pi\lambda$ $(0 \leqslant \lambda < 2)$ can be described in terms of the solution of the differential equation

$$y''' + yy'' + \lambda(1-y'^2) = 0, \tag{1}$$

which satisfies the boundary conditions

$$y = y' = 0 \quad \text{at} \quad x = 0, \quad y' \to 1 \quad \text{as} \quad x \to \infty. \tag{2}$$

The special case $\lambda = 0$, in which the wedge reduces to a flat plate, was solved numerically by Blasius (1908) and is the subject of an extensive literature. The general case was first discussed by Falkner & Skan (1931). Their numerical calculations were afterwards extended by Hartree (1937). A rigorous proof of the solubility of the boundary-value problem (1) and (2), for any non-negative constant λ, has been given by Weyl (1942).† Weyl showed that the problem has a solution $y(x)$ whose first derivative increases with x and whose second derivative decreases and tends to zero as $x \to \infty$. His proof depended on the theory of fixed points of functional operators and a skilful use of inequalities, and was consequently neither constructive nor simple.

† Weyl's equation (A_λ) is obtained from (1) by putting $w = (\frac{1}{2}k)^{\frac{1}{2}}\, y$, $z = (2k)^{-\frac{1}{2}}\, x$.

The purpose of the present paper is to study the equation (1) from the standpoint of the theory of differential equations. In § 2 a direct and elementary proof of Weyl's theorem is given for all $\lambda > 0$. Moreover, the same proof shows that the problem is still soluble if the boundary conditions (2) are replaced by the more general ones

$$y = \alpha, \quad y' = \beta \quad \text{at} \quad x = 0; \qquad y' \to 1 \quad \text{as} \quad x \to \infty, \tag{2$'$}$$

where α and β are arbitrary non-negative constants. It is further shown that the solution is unique in a certain sense.

Section 3 is concerned with obtaining bounds for the value of $y''(0)$ which characterizes the solution of the problem (1) and (2)$'$. For the boundary conditions (2) it is shown in particular that

$$\tfrac{4}{3}\lambda < [y''(0)]^2 < \tfrac{4}{3}\lambda + \tfrac{1}{3}.$$

These bounds are much more exact and more explicit than those obtained by Weyl. It is also proved that $y''(0)$ is an increasing function of λ.

In § 4 the asymptotic theory of linear differential equations is used to obtain the precise form for $x \to \infty$ of the solution of the boundary-value problem. The result, which is expressed in terms of parabolic cylinder functions, provides a justification of the method of numerical integration used by Blasius for $\lambda = 0$.

The remainder of the paper is devoted to a study of *all* solutions of the differential equation (1). It is proposed to classify the solutions according to their limit-sets and to obtain the asymptotic form of each type.

This programme is carried through completely for $0 < \lambda \leqslant \frac{1}{2}$. Apart from the solutions for which $y' \equiv -1$ there are four classes of solution whose asymptotic behaviour is distinguished by

(i) $y' \to 1$,

(ii) $y' \sim a y^{\lambda} \quad (a > 0)$,

(iii) $y' \sim a\,|y|^{\lambda} \quad (a < 0)$,

(iv) $y' \sim \frac{1}{6}(\lambda - 2)\, y^2$.

The solutions of types (ii) and (iv) both form 3-parameter manifolds. The solutions of types (i) and (iii) form 2-parameter manifolds. It is remarkable that the second derivative of any solution vanishes at most once, unless it vanishes identically.

The discussion for $\lambda > \frac{1}{2}$ is more difficult and lacks completeness in some respects. However, it is shown that solutions of types (ii) and (iii) exist for all values of λ less than 2. On the other hand, it is shown that all non-oscillatory solutions are of types (i) and (iv) if $\lambda > 2$.

The value $\lambda = 0$ is exceptional in some ways and is discussed separately. The results for this case are summarized at the end of § 9.

This last part of the investigation may perhaps throw some light on the underlying assumptions of boundary-layer theory and the extent to which these assumptions are satisfied in practice. A pointer in this direction is the fact that the form of the solutions for $0 < \lambda < 2$, i.e. for the range of values suggested by the hydrodynamical application, is very different from their form for $\lambda > 2$. The boundary-value problem, however, makes no distinction between values of λ less than and greater than 2. In any case, the rather complete results obtained should do something to dispel the bogey of non-linearity and may serve as a model for the discussion of other problems.

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2. Solution of the boundary-value problem for $\lambda > 0$

We replace the equation (1) by the equivalent system

$$\left.\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ y_3' &= -y_1 y_3 - \lambda(1-y_2^2). \end{aligned}\right\} \tag{3}$$

Since this system is autonomous we can represent its solutions by curves in the three-dimensional phase space of the variables (y_1, y_2, y_3), with x as curve parameter. Two solutions which differ only by a displacement of the origin of x are represented by the same curve or *path*. By the usual theorems on the existence and uniqueness of solutions one and only one path passes through each point, and any finite arc of such a path depends continuously on the initial point. Moreover, if $\mathbf{y} = \mathbf{y}(x)$ is the solution of (3) which takes the value $\mathbf{y}^{(0)} = (y_1^{(0)}, y_2^{(0)}, y_3^{(0)})$ at $x = 0$ it is defined either for all $x \geqslant 0$ or else over an interval $0 \leqslant x < x^*$ such that $\|\mathbf{y}\| = |y_1| + |y_2| + |y_3|$ tends to infinity as $x \to x^*$. This follows immediately from well-known results on the continuation of solutions.

Consider now the path C which passes through the point (α, β, γ), where $\alpha \geqslant 0$, $0 \leqslant \beta < 1$, $\gamma > 0$. We imagine C described in the sense of increasing x, starting from this point. It follows from the differential equations (3) that C passes into the domain D defined by the inequalities $y_1 > 0$, $0 < y_2 < 1$, $y_3 > 0$. On any path in this domain y_1 and y_2 are increasing functions of x and y_3 is a decreasing function. No path can tend to infinity in D for a finite value of x. For on any path in D y_2 and y_3 are bounded. Thus y_1' is bounded, and hence also y_1 over any finite interval. Finally, no path can leave D through the edge $y_1 > 0$, $y_2 = 1$, $y_3 = 0$, since the solution which takes the value $(\alpha, 1, 0)$ at $x = 0$ is given by

$$y_1 \equiv x + \alpha, \quad y_2 \equiv 1, \quad y_3 \equiv 0.$$

This simple fact, which does not seem to have been noticed before, forms the basis of our proof.

It follows that there are just three possibilities:

(*a*) C leaves D through the face $y_1 > 0$, $y_2 = 1$, $y_3 > 0$,

(*b*) C leaves D through the face $y_1 > 0$, $0 < y_2 < 1$, $y_3 = 0$,

(*c*) C is defined and remains in D for all $x > 0$.

Let us consider the last possibility more closely. Since $y_2' = y_3$ is a positive, decreasing function and y_2 remains bounded we must have $y_3 \to 0$ as $x \to \infty$. Similarly, since y_2 is an increasing function, since y_3 remains bounded, and $y_3' < -\lambda(1-y_2^2)$ we must have $y_2 \to 1$ as $x \to \infty$. Since $y_1' = y_2$ this implies that $y_1 \to \infty$. Thus

$$y_1 \to \infty, \quad y_2 \to 1, \quad y_3 \to 0 \quad \text{as} \quad x \to \infty.$$

Suppose now that α and β are held fixed but γ is allowed to vary, so that $C = C(\gamma)$ becomes a function of γ. From the initial conditions

$$y_1 = \alpha, \quad y_2 = \beta, \quad y_3 = \gamma,$$

$$y_1' = \beta, \quad y_2' = \gamma, \quad y_3' = -\alpha\gamma - \lambda(1-\beta^2),$$

it is evident that $C(\gamma)$ is of type (b) for all sufficiently small γ. In fact if x is small then y_2' is small, but y_3' is nearly equal to the negative quantity $-\lambda(1-\beta^2)$. We shall now show that $C(\gamma)$ is of type (a) for all sufficiently large γ. We have

$$\begin{aligned} y_3' &= -(y_1 y_2)' + y_2^2 - \lambda(1-y_2^2) \\ &\geqslant -(y_1 y_2)' + \beta^2 - \lambda(1-\beta^2). \end{aligned}$$

Integrating, we get

$$y_3(x) \geqslant \gamma + \alpha\beta - y_1(x)\, y_2(x) + \beta^2 x - \lambda(1-\beta^2)\, x.$$

But $y_2(x) < 1$ in D and hence $y_1(x) < \alpha + x$. Therefore, for as long as C remains in D,

$$y_3(x) \geqslant \gamma + \alpha\beta - \alpha - (\lambda+1)\,(1-\beta^2)\, x.$$

It follows that if γ is sufficiently large $C(\gamma)$ leaves D through the face $y_2 = 1$.

The values of γ for which $C(\gamma)$ is of type (a) or type (b) form open subsets of the half-line $0 < \gamma < \infty$. For in the first case $C(\gamma)$ passes into the domain $y_1 > 0$, $y_2 > 1$, $y_3 > 0$ and in the second case it passes into the domain $y_1 > 0$, $0 < y_2 < 1$, $y_3 < 0$. Since the half-line $0 < \gamma < \infty$ is connected it follows that $C(\gamma)$ must be of type (c) for at least one value of γ. Thus we have proved that the boundary-value problem (1) and (2)$'$, where $\alpha \geqslant 0$ and $0 \leqslant \beta < 1$, has a solution $y = y(x)$ for which y'' is always positive. In particular, for $\alpha = \beta = 0$, we obtain Weyl's result.

We shall now show that the boundary-value problem has only one solution for which y'' is always positive. In other words, we shall show that there is only one value of γ in the interval $0 < \gamma < \infty$ for which $C(\gamma)$ is of type (c). Since the differential equation (1) is autonomous its order can be reduced by considering $y' = z$ as a function of y. We have

$$y'' = z\,\mathrm{d}z/\mathrm{d}y, \quad y''' = z(\mathrm{d}z/\mathrm{d}y)^2 + z^2\,\mathrm{d}^2z/\mathrm{d}y^2,$$

so that (1) is replaced by the second-order equation

$$\frac{\mathrm{d}^2z}{\mathrm{d}y^2} = -\frac{1}{z}\left(\frac{\mathrm{d}z}{\mathrm{d}y}\right)^2 - \frac{y}{z}\frac{\mathrm{d}z}{\mathrm{d}y} + \lambda\left(1-\frac{1}{z^2}\right). \tag{4}$$

The right-hand side is an increasing function of z for $z > 0$, $y\,\mathrm{d}z/\mathrm{d}y \geqslant 0$. We wish to show that there is at most one solution of (4) which remains in the domain $z > 0$, $\mathrm{d}z/\mathrm{d}y > 0$ for $\alpha < y < \infty$ and which has prescribed limits $z \to \beta$ as $y \to \alpha$, $z \to 1$ as $y \to \infty$. This, however, is a consequence of the following theorem (cf. Sansone 1949):

Let $z(x)$ and $w(x)$ be continuous for $a \leqslant x \leqslant b$ and satisfy the differential inequalities

$$z'' \leqslant F(x, z, z'), \quad w'' \geqslant F(x, w, w')$$

for $a < x < b$, where F is an increasing function of its middle argument.

If $w(a) \leqslant z(a)$ and $w(b) \leqslant z(b)$ then $w(x) \leqslant z(x)$ for $a \leqslant x \leqslant b$.

Proof. Otherwise, if $w(x) > z(x)$ for some value of x, the function $v(x) = w(x) - z(x)$ would have a positive maximum at a point c between a and b. Then $v(c) > 0$, $v'(c) = 0$, and $v''(c) \leqslant 0$. But, since $w(c) > z(c)$ and $w'(c) = z'(c)$,

$$v''(c) \geqslant F[c, w(c), w'(c)] - F[c, z(c), z'(c)] > 0,$$

which is a contradiction.

In particular, two solutions of the differential equation $z'' = F(x, z, z')$ which take the same values at a and b must coincide throughout the interval $a \leqslant x \leqslant b$.

In the same way we may consider the path C which passes through the point (α, β, γ), where $\alpha \geqslant 0$, $\beta > 1$ and $\gamma < 0$. This path passes into the domain D defined by the inequalities $y_1 > 0$, $y_2 > 1$, $y_3 < 0$. On any path in this domain y_1 and y_3 are increasing functions and y_2 is a decreasing function. It follows as before that there are just three possibilities:

(*a*) C leaves D through the face $y_1 > 0$, $y_2 = 1$, $y_3 < 0$,
(*b*) C leaves D through the face $y_1 > 0$, $y_2 > 1$, $y_3 = 0$,
(*c*) C is defined and remains in D for all $x > 0$.

In the last case $y_1 \to \infty$, $y_2 \to 1$, and $y_3 \to 0$ as $x \to \infty$.

Let α and β be held fixed and let γ vary in the interval $-\infty < \gamma < 0$. It is easily seen from the initial conditions that C is of type (*b*) if $|\gamma|$ is sufficiently small. Furthermore, C is of type (*a*) if $|\gamma|$ is sufficiently large. In fact the same argument as before establishes the inequality

$$y_3(x) \leqslant \gamma + \alpha\beta - \alpha + (\lambda+1)(\beta^2 - 1)\,x,$$

for as long as C remains in D. Since the values of γ for which C is of type (*a*) or (*b*) form open subsets of the half-line $-\infty < \gamma < 0$ it follows that C is of type (*c*) for at least one value of γ.

We can again show that there is only one value of γ for which C is of type (*c*), but a different method of proof is required. We depend this time on the following theorem, which is a special case of a theorem due to Kamke (1932).

Let $F(x, y_1, y_2, \ldots, y_n)$ be continuous in some domain of $(n+1)$-dimensional space and a non-decreasing function of the variables $(y_1, \ldots, y_{n-1})$. Suppose further that through each point there passes only one solution of the differential equation

$$w^{(n)} = F(x, w, w', \ldots, w^{(n-1)}).$$

Let $w(x)$ be a solution of this differential equation and let $z(x)$ be a solution of the differential inequality

$$z^{(n)} \leqslant F(x, z, z', \ldots, z^{(n-1)})$$

in an interval $a \leqslant x < b$. Under these conditions if

$$z^{(i)}(a) \leqslant w^{(i)}(a) \quad (i = 0, \ldots, n-1) \qquad \textit{then} \qquad z^{(i)}(x) \leqslant w^{(i)}(x) \quad (i = 0, \ldots, n-1)$$

for $a < x < b$.

The theorem still holds when the inequalities for z are all reversed. This theorem will be of constant service to us in the following and will be referred to as 'Kamke's theorem'. For our purposes it is important to have conditions which ensure that $z(x) \equiv w(x)$. Suppose first that $n > 1$ and that $w(x)$ and $z(x)$ have the same finite limit as $x \to b$. Since $w'(x) \geqslant z'(x)$ for $a < x < b$ the difference $w(x) - z(x)$ is a non-decreasing function. But $w(x) - z(x)$ is non-negative for $x = a$ and tends to zero as $x \to b$. Therefore it is identically zero. Suppose next that $n = 1$, that $w(x)$ and $z(x)$ have the same finite limit l as $x \to b$, and that the point (b, l) lies in the domain of definition of the function F. Then by Kamke's theorem itself, with x replaced by $-x$, $z(b) = w(b)$ implies $z(x) \geqslant w(x)$ for $a \leqslant x \leqslant b$. Combining this with the previous inequality we obtain $z(x) \equiv w(x)$.†

To apply this theorem to the differential equation (1) we take

$$F(x, y_1, y_2, y_3) = -y_1 y_3 + \lambda(y_2^2 - 1).$$

† The additional assumption for $n = 1$ is easily seen to be necessary.

Evidently F is a non-decreasing function of (y_1, y_2) in the region $y_3 \leqslant 0$, $y_2 \geqslant 0$. If there were two solutions $y(x)$ and $\bar{y}(x)$ of type (c), corresponding to the values γ and $\bar{\gamma}$, respectively, where $\bar{\gamma} < \gamma$, then by Kamke's theorem we would have $\bar{y}^{(i)}(x) \leqslant y^{(i)}(x)$ $(i = 0, 1, 2)$ for all $x \geqslant 0$. Since $y'(x)$ and $\bar{y}'(x)$ have the same finite limit for $x \to \infty$ this is only possible if $\bar{y}'(x) \equiv y'(x)$. But $y'(x) - \bar{y}'(x)$ is positive for small values of x, because $y''(0) > \bar{y}''(0)$. Thus we have a contradiction.

Taken together, our results for $\beta < 1$ and $\beta > 1$ establish the following

THEOREM. *The boundary-value problem*

$$y''' + yy'' + \lambda(1 - y'^2) = 0,$$

$$y = \alpha, \quad y' = \beta \quad \text{at} \quad x = 0; \qquad y' \to 1 \quad \text{as} \quad x \to \infty$$

has a solution for any non-negative values of the constants α, β. *The second derivative* y'' *is positive, zero, or negative throughout the interval* $0 \leqslant x < \infty$ *according as* β *is less than, equal to, or greater than* 1. *Moreover, with this restriction on* y'' *the solution of the problem is unique.*

3. Dependence of the solution on the parameter λ

To avoid case distinctions we will suppose throughout this section and the next that $0 \leqslant \beta < 1$. However, this is a restriction of convenience rather than necessity. We consider first the problem of obtaining bounds for the value of $y''(0)$ which characterizes the solution of the boundary-value problem. In the hydrodynamical problem the initial value of y'' determines the *skin friction* on the wedge.

Let $y(x)$ be the solution of the boundary problem (1) and (2)$'$. Since $yy'' \geqslant 0$ we have $y''' \leqslant \lambda(y'^2 - 1)$. Thus $z(x) = y'(x)$ satisfies the differential inequality

$$z'' \leqslant \lambda(z^2 - 1).$$

The function

$$w(x) = 3 \tanh^2 (\tfrac{1}{2}\lambda)^{\frac{1}{2}} (x + x_0) - 2$$

is a solution of the differential equation

$$w'' = \lambda(w^2 - 1)$$

and tends to 1 as $x \to \infty$. We can choose x_0 so that $w(0) = \beta = z(0)$. If $z'(0) \leqslant w'(0)$ then by Kamke's theorem, with $n = 2$, $z(x) \leqslant w(x)$ for all $x \geqslant 0$. Moreover, since $z(x)$ and $w(x)$ have the same finite limit as $x \to \infty$ we must actually have $z(x) \equiv w(x)$. But this is impossible, because $yy'' > 0$ for $x > 0$. Therefore $z'(0) > w'(0)$. From the definition of w and the initial condition $w(0) = \beta$ we find that

$$w'(0) = (\tfrac{2}{3}\lambda)^{\frac{1}{2}} (2 + \beta)^{\frac{1}{2}} (1 - \beta).$$

Hence *the initial value* γ *of* y'' *which corresponds to the solution of the boundary-value problem satisfies the inequality*

$$\gamma^2 > \tfrac{2}{3}\lambda(2 + \beta)(1 - \beta)^2.$$

Bounds in the opposite direction can be obtained in the same manner. For simplicity we will only consider the case $\alpha = 0$. The function $\frac{1}{2}y'^2 - yy''$ has as its derivative $-yy'''$ and consequently is an increasing function of x. Therefore $\frac{1}{2}y'^2 \geqslant yy''$ for $x \geqslant 0$. It follows that $z(x) = y'(x)$ satisfies the differential inequality

$$z'' \geqslant (\lambda - \tfrac{1}{2}) z^2 - \lambda.$$

If $\lambda \geqslant \frac{1}{2}$ we can compare this differential inequality with the differential equation

$$w'' = (\lambda - \tfrac{1}{2})\, w^2 - \lambda. \tag{5}$$

Put $w' = v$, so that $w'' = v\,\mathrm{d}v/\mathrm{d}w$. Then (5) has the first integral

$$v^2 = \tfrac{1}{3}(2\lambda - 1)\, w^3 - 2\lambda w + \text{constant}.$$

If $v = 0$ when $w = 1$ then

$$\begin{aligned} v^2 &= \tfrac{1}{3}(2\lambda-1)\,(w^3-1) - 2\lambda(w-1) \\ &= \tfrac{2}{3}\lambda(2+w)\,(1-w)^2 + \tfrac{1}{3}(1-w^3) \\ &= C(w), \quad \text{say.} \end{aligned}$$

The cubic $C(w)$ has a simple zero at $w = 1$ and is positive in the interval $0 \leqslant w < 1$. Hence the function $w(x)$ defined by the equation

$$x = \int_\beta^w \frac{\mathrm{d}w}{[C(w)]^{\frac{1}{2}}}$$

is a solution of (5) for $0 \leqslant x < x^*$, where

$$x^* = \int_\beta^1 \frac{\mathrm{d}w}{[C(w)]^{\frac{1}{2}}},$$

and satisfies the boundary conditions $w = \beta$ at $x = 0$, $w \to 1$ as $x \to x^*$. If $z'(0) \geqslant w'(0)$ then by Kamke's theorem $z(x^*) \geqslant 1$, which is impossible. Therefore $z'(0) < w'(0)$. Since $w'(0) = [C(\beta)]^{\frac{1}{2}}$ this gives us the following inequality:

If $\alpha = 0$ and $\lambda \geqslant \frac{1}{2}$ the initial value γ of y'' which corresponds to the solution of the boundary-value problem satisfies the inequality

$$\gamma^2 < \tfrac{2}{3}\lambda(2+\beta)\,(1-\beta)^2 + \tfrac{1}{3}(1-\beta^3).\dagger$$

For the problem considered by Weyl these inequalities reduce to

$$\tfrac{4}{3}\lambda < \gamma^2 < \tfrac{4}{3}\lambda + \tfrac{1}{3},$$

the inequality on the right being established only for $\lambda \geqslant \frac{1}{2}$. Having regard to the nature of the problem and the ease with which they have been obtained, these bounds are very satisfactory. They show in particular that $\gamma \sim (\frac{4}{3}\lambda)^{\frac{1}{2}}$ for $\lambda \to \infty$.

Similarly, we can find upper bounds for γ^2 when $0 \leqslant \lambda < \frac{1}{2}$. However, I will only show how the rather precise bound obtained by Weyl in the special case $\lambda = 0$ may be recovered by the present method. Suppose $\alpha = \beta = \lambda = 0$. From the inequality $y'' \leqslant \gamma$ for $x \geqslant 0$ we obtain by integration $y \leqslant \frac{1}{2}\gamma x^2$. It follows from (1), with $\lambda = 0$, that $y''' \geqslant -\frac{1}{2}\gamma x^2 y''$. Thus $z(x) = y'(x)$ satisfies the differential inequality

$$z'' \geqslant -\tfrac{1}{2}\gamma x^2 z'.$$

The function

$$w(x) = A\int_0^x \mathrm{e}^{-\frac{1}{6}\gamma\xi^3}\,\mathrm{d}\xi$$

is a solution of the differential equation $w'' = -\frac{1}{2}\gamma x^2 w'$ and $\to 1$ as $x \to \infty$ if we choose

$$A = \left[\int_0^\infty \mathrm{e}^{-\frac{1}{6}\gamma x^3}\,\mathrm{d}x\right]^{-1}.$$

† *Added in proof* (*July*, 1960). The restriction $\lambda \geqslant \frac{1}{2}$ may be removed, the proof much simplified, and the bound itself sharpened to

$$\gamma^2 < \tfrac{1}{3}(1-\beta)^2\,[2\lambda(2+\beta) + 1 + 2\beta]$$

by putting $u = z'$ and integrating the first-order differential inequality

$$u\,\mathrm{d}u/\mathrm{d}z > (\lambda - \tfrac{1}{2})\, z^2 - \lambda + \tfrac{1}{2}\beta^2.$$

The proof of the lower bound for γ^2 may be simplified similarly.

Moreover, $w = 0$ at $x = 0$. We must therefore have $z'(0) < w'(0)$. But $w'(0) = A$. Hence

$$\gamma^{-1} > A^{-1} = (2/\gamma)^{\frac{1}{3}} \int_0^\infty e^{-\frac{1}{3}x^3}\, dx$$

or

$$\gamma^{-\frac{2}{3}} > 2^{\frac{1}{3}} \int_0^\infty e^{-\frac{1}{3}x^3}\, dx.$$

Except for notation this is the same as the bound obtained by Weyl (1942, p. 388).

We turn next to the investigation of the solution itself as a function of λ. Let $\bar{y}(x)$ be the solution of the boundary-value problem with λ replaced by $\bar{\lambda}$, where $\bar{\lambda} > \lambda$. Then $\bar{z} = \bar{y}'$ is a solution of (4) with λ replaced by $\bar{\lambda}$. Thus $\bar{z}$ satisfies the differential inequality

$$\frac{d^2z}{dy^2} < -\frac{1}{z}\left(\frac{dz}{dy}\right)^2 - \frac{y}{z}\frac{dz}{dy} + \lambda\left(1 - \frac{1}{z^2}\right).$$

Moreover, $\bar{z}$ and z have the same limiting values for $y \to \alpha$ and $y \to \infty$. Therefore, by the first theorem on differential inequalities which was used in § 2, $\bar{z}(y) > z(y)$ for $\alpha < y < \infty$.†
Hence

$$[\bar{z}(y)]^{-1} < [z(y)]^{-1} \quad \text{for} \quad y > \alpha.$$

Integrating both sides with respect to y between the limits α and η we get the inequality $\bar{x}(\eta) < x(\eta)$ between the functions inverse to $\bar{y}(x)$ and $y(x)$. Since

$$\bar{y}[\bar{x}(\eta)] = \eta = y[x(\eta)]$$

and y is an increasing function of x it follows that $\bar{y}(x) > y(x)$ for $0 < x < \infty$. Since $\bar{z}$ is an increasing function of x it further follows that $\bar{z}(x) > z(x)$. Thus *the solution of the boundary-value problem, and also its first derivative, are increasing functions of λ for each fixed value of x.*

We can draw from this a further conclusion regarding the initial value of y''. Since $\bar{y} = y$ and $\bar{y}' = y'$ at $x = 0$ we must have $\bar{y}'' \geq y''$ at $x = 0$. Moreover, if $\bar{y}'' = y''$ we must also have $\bar{y}''' \geq y'''$. However, this contradicts the differential equation (1). Therefore equality is impossible and *the initial value of the second derivative is likewise an increasing function of λ.*

4. Asymptotic form of the solution for large values of x

The solution $y(x)$ of the boundary-value problem (1) and (2)′ satisfies the differential inequality $y''' + yy'' \leq 0$. If θ is any positive number less than 1 and if we choose x_0 so large that $y'(x_0) \geq \theta$ then $y(x) \geq \theta(x - x_0)$ for $x \geq x_0$. Consequently

$$y''' \leq -\theta(x - x_0)\, y'' \quad \text{for} \quad x \geq x_0.$$

Dividing by y'' and integrating with respect to x we get

$$y''(x) \leq y''(x_0)\, e^{-\frac{1}{2}\theta(x-x_0)^2}.$$

Thus

$$y''(x) = \mathcal{O}(e^{-\frac{1}{2}x^2}) \quad \text{for} \quad x \to \infty.$$

Here the notation $w = \mathcal{O}(z)$ is used to mean that $w = O(z^{1-\epsilon})$ for arbitrarily small positive values of ϵ. It follows that

$$y'(x) = 1 + \mathcal{O}(e^{-\frac{1}{2}x^2}),$$
$$y(x) = x - \delta + \mathcal{O}(e^{-\frac{1}{2}x^2}),$$

where δ is a constant:

$$\delta = -\alpha + \int_0^\infty (1 - y')\, dx. \tag{6}$$

† Since we now have strict inequality in the hypothesis of the theorem we also have strict inequality in the conclusion.

In the hydrodynamical problem the quantity δ defined by (6) is called the *displacement thickness* and is a measure of the thickness of the boundary layer.

In order to obtain more precise asymptotic expressions we rewrite the differential equation (1) in the form

$$y'''+xy''+2\lambda(1-y') = (x-y)\,y''+\lambda(1-y')^2.$$

If we put $Y(x) = x-y(x+\delta)$ it follows that $w(x) = Y'(x)$ is a solution of the differential equation

$$w''+xw'-2\lambda w = f(x), \tag{7}$$

where

$$f(x) = Y(x)\,Y''(x)-\lambda[Y'(x)]^2. \tag{8}$$

The estimates already obtained for $y(x)$ show that $f(x) = \mathcal{O}(e^{-x^2})$ for $x\to\infty$. Since $f(x)$ is small for large x the equation (7) is in a convenient form for the application of the powerful method which Liouville first used to obtain asymptotic expressions for the eigenfunctions of second-order linear differential equations.

The homogeneous linear differential equation

$$w''+xw'-2\lambda w = 0$$

has the fundamental system of solutions

$$w_1(x) = e^{-\frac{1}{4}x^2}\,D(x),$$

$$w_2(x) = e^{-\frac{1}{4}x^2}\,D(-x),$$

where $D(x)$ denotes the parabolic cylinder function $D_\nu(x)$ of order $\nu = -1-2\lambda$. A detailed account of these functions is given by Erdélyi (1953) and they have recently been tabulated (Miller 1955).

The solutions of the inhomogeneous equation (7) are given, according to the variation of constants formula, by

$$w = c_1 w_1(x)+c_2 w_2(x)+\int_{x_0}^{x} f(\xi)\,[w_1(\xi)\,w_2(x)-w_1(x)\,w_2(\xi)]\,d\xi/W(\xi),$$

where c_1 and c_2 are arbitrary constants and

$$W = w_1 w_2'-w_1' w_2$$

is the Wronskian of w_1 and w_2. From the known Wronskian of the parabolic cylinder functions it follows that

$$W(\xi) = \frac{(2\pi)^{\frac{1}{2}}}{(2\lambda)!}\,e^{-\frac{1}{2}\xi^2}.$$

The asymptotic expansions of the parabolic cylinder functions show also that for $x\to+\infty$

$$w_1(x) \sim x^{-1-2\lambda}\,e^{-\frac{1}{2}x^2},$$

$$w_2(x) \sim \frac{(2\pi)^{\frac{1}{2}}}{(2\lambda)!}\,x^{2\lambda}.$$

Hence if $w\to 0$ as $x\to\infty$ then

$$w = cw_1(x)-\int_{x}^{\infty} f(\xi)\,[w_1(\xi)\,w_2(x)-w_1(x)\,w_2(\xi)]\,d\xi/W(\xi),$$

provided the integral on the right converges. But this integral certainly converges when $f(x)$ is given by (8) and is actually of order $\mathcal{O}(e^{-x^2})$. The required asymptotic expression now follows immediately:

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The derivative of the solution of the boundary-value problem has for $x \to \infty$ the form

$$y'(x+\delta) = 1 - c\,e^{-\frac{1}{4}x^2} D_{-1-2\lambda}(x) + \mathcal{O}(e^{-x^2}), \tag{8a}$$

where c and δ are constants. Less precisely,

$$1 - y'(x+\delta) \sim cx^{-1-2\lambda} e^{-\frac{1}{2}x^2}. \tag{8b}$$

If we put
$$\phi_\nu(x) = e^{-\frac{1}{4}x^2} D_{-\nu}(x)$$
then
$$\phi'_\nu(x) = -\phi_{\nu-1}(x).$$
Consequently the asymptotic form of y is given by

$$y(x+\delta) = x + c\,e^{-\frac{1}{4}x^2} D_{-2-2\lambda}(x) + \mathcal{O}(e^{-x^2}).$$

By virtue of the differential equation (1) the asymptotic expression for y' can also be differentiated:
$$y''(x+\delta) = c\,e^{-\frac{1}{4}x^2} D_{-2\lambda}(x) + \mathcal{O}(e^{-x^2}).$$

The second-order approximation found by Blasius in the case $\lambda = 0$ can be obtained very simply, and rigorously, by the present method. We have only to replace w by cw_1 on the right-hand side of the integral equation for w. Since $w_1 = \phi_1$ and $w_2 = (2\pi)^{\frac{1}{2}} - \phi_1$ when $\lambda = 0$, the integral is replaced by

$$-c^2 \int_x^\infty \phi_0(\xi)\,\phi_2(\xi)\,[\phi_1(\xi) - \phi_1(x)]\,d\xi/\phi_0(\xi) = -c^2[\tfrac{1}{2}\phi_2^2(x) - \phi_1(x)\,\phi_3(x)].$$

Therefore *for* $\lambda = 0$

$$y'(x+\delta) \approx 1 - c\,e^{-\frac{1}{4}x^2} D_{-1}(x) - c^2 e^{-\frac{1}{2}x^2}[D_{-1}(x)\,D_{-3}(x) - \tfrac{1}{2}D^2_{-2}(x)].$$

We have supposed $0 \leqslant \beta < 1$, but matters are even simpler when $\beta > 1$. For in this case the solution $y(x)$ of the boundary-value problem (1) and (2)$'$ satisfies the differential inequality $y''' + xy'' \geqslant 0$. Therefore

$$0 > y''(x) \geqslant \gamma\,e^{-\frac{1}{2}x^2} \quad \text{for} \quad x \geqslant 0.$$

From this point on the argument is the same as before.

It may be shown by the method of successive approximations that the integro-differential equation
$$Y'(x) = cw_1(x) - \int_x^\infty f(\xi)\,[w_1(\xi)\,w_2(x) - w_1(x)\,w_2(\xi)]\,d\xi/W(\xi),$$

where $f(x) = YY'' - \lambda Y'^2$, has a unique solution defined for all sufficiently large x for which

$$Y'' = \mathcal{O}(e^{-\frac{1}{2}x^2}), \quad Y \to 0 \quad \text{as} \quad x \to \infty.\dagger$$

Therefore, *for any given constants c and δ, the differential equation* (1) *has a unique solution $y(x)$ defined for all sufficiently large x such that*

$$1 - y'(x+\delta) \sim cx^{-1-2\lambda} e^{-\frac{1}{2}x^2}.$$

5. General properties of the boundary-layer equation

The hydrodynamical problem has already been generalized by allowing the initial values of y and y' to be arbitrary non-negative numbers. It is natural now to go one step further and inquire into the totality of solutions of the differential equation (1).

† The argument for a similar, but simpler, case is given in full in §9.

Since equation (1) is analytic its solutions are analytic functions of x. Moreover, since it is autonomous, $y = \phi(x+x_0)$ is a solution at the same time as $y = \phi(x)$. The equation also has two useful symmetry properties:

THEOREM 1. *If* $y = \phi(x)$ *is a solution of* (1) *then* $y = -\phi(-x)$ *and* $y = \mathrm{i}\phi(\mathrm{i}x)$ *are also solutions.* ($\mathrm{i}^2 = -1$.)

In fact the differential equation is invariant under the transformation $y \to hy$, $x \to h^{-1}x$ if $h^4 = 1$. It follows from the first property that when studying the behaviour of the solutions in the real domain we need only consider non-negative values of x.

If $y = y'' = 0$ at $x = 0$ then y is an odd function of x, since the solutions $\phi(x)$ and $-\phi(-x)$ satisfy the same initial conditions. We can therefore write $y = x\psi(x^2)$, where ψ is an analytic function of x^2. If $y = y' = y'' = 0$ at $x = 0$ we can actually write $y = x^3\chi(x^4)$, where χ is an analytic function of x^4. For in this case the solutions $\phi(x)$ and $\mathrm{i}\phi(\mathrm{i}x)$ satisfy the same initial conditions. Therefore $\phi(x) = \mathrm{i}\phi(\mathrm{i}x)$, which implies $\psi(x^2) = -\psi(-x^2)$. Hence $\psi(x^2)$ has the form $x^2\chi(x^4)$.

Again, *if* $y'' = y''' = 0$ *at* $x = 0$ *then* y'' *vanishes identically*. For any such solution is uniquely determined by the values of y and y' at $x = 0$. If $\lambda \neq 0$ the differential equation (1) shows that $y'^2 = 1$ at $x = 0$. But, as we saw in § 2, $y = \alpha \pm x$ is a solution for any value of the constant α. On the other hand, if $\lambda = 0$, $y = \alpha + \beta x$ is a solution for any values of the constants α, β.

THEOREM 2. *Let* $y = y(x)$ *be a solution of the differential equation* (1) *for all large* x. *If* y' *converges to a finite limit as* $x \to \infty$ *this limit must be* ± 1, *unless* $\lambda = 0$. *If* y'' *converges to a finite limit as* $x \to \infty$ *this limit must be* 0, *unless* $\lambda = \frac{1}{2}$. *If* y''' *converges to a finite limit as* $x \to \infty$ *this limit must be* 0, *unless* $\lambda = \frac{2}{3}$.

Proof. Integrating (1) by parts between the limits 0 and x we get

$$y''(x) - y''(0) + y(x)\,y'(x) - y(0)\,y'(0) + \lambda x = (\lambda+1)\int_0^x y'^2\,\mathrm{d}x. \tag{9}$$

If $y' \to \mu$ as $x \to \infty$ then by the theory of indeterminate forms

$$y \sim \mu x, \quad \int_0^x y'^2\,\mathrm{d}x \sim \mu^2 x.$$

Substituting these expressions in (9) we get

$$y'' + (\mu^2+\lambda)\,x = (\lambda+1)\,\mu^2 x + o(x),$$

or $y'' \sim \lambda(\mu^2-1)\,x$. Since y' remains bounded for $x \to \infty$ this is only possible if $\lambda = 0$ or $\mu^2 = 1$.

Similarly if $y'' \to \nu$ as $x \to \infty$ then $y' \sim \nu x$ and $y \sim \frac{1}{2}\nu x^2$. Substituting these expressions in (1) we get

$$y''' + (\tfrac{1}{2} - \lambda)\,\nu^2 x^2 = o(x^2),$$

or $y''' \sim (\lambda - \frac{1}{2})\,\nu^2 x^2$. Hence $y'' \sim \frac{1}{3}(\lambda-\frac{1}{2})\,\nu^2 x^3$. This contradicts the hypothesis $y'' \to \nu$ unless $\lambda = \frac{1}{2}$ or $\nu = 0$. The last part of the theorem is proved in exactly the same way.

The provisos in the statement of this theorem are all necessary since $y = ax$ *is a solution for any constant* a *when* $\lambda = 0$, $y = (x^2 - b^2)/2b$ *is a solution for any constant* $b \neq 0$ *when* $\lambda = \frac{1}{2}$, *and* $y = -\frac{1}{9}x^3$ *is a solution when* $\lambda = \frac{2}{3}$.

The argument used to prove theorem 2 can also be applied to the differential equation (4).

Theorem 3. *If $z = z(y)$ is a solution of (4) for all large y and $z \sim \mu y^{\alpha}$ for $y \to +\infty$, where $\mu \neq 0$ and $\alpha > 0$, then either $\alpha < 2$ and $\lambda = \alpha$ or $\alpha = 2$ and $\mu = \frac{1}{6}(\lambda - 2)$.*

Proof. Writing (4) in the form

$$\mathrm{d}(zz' + yz)/\mathrm{d}y = (\lambda+1)\,z - \lambda z^{-1}$$

we get by integration

$$zz' + yz \sim (\lambda+1)\,\mu y^{\alpha+1}/(\alpha+1).$$

Hence

$$z' \sim \left(\frac{\lambda+1}{\alpha+1} - 1\right) y$$

and

$$z \sim \frac{1}{2}\left(\frac{\lambda+1}{\alpha+1} - 1\right) y^2.$$

This contradicts the original hypothesis if $\alpha > 2$. If $\alpha = 2$ we again obtain a contradiction unless $\mu = \frac{1}{2}[\frac{1}{3}(\lambda+1) - 1] = \frac{1}{6}(\lambda - 2)$. If $\alpha < 2$ we obtain a contradiction unless $\lambda = \alpha$.

The proof shows also that if $y^{-2}z \to \frac{1}{6}(\lambda-2)$ then $y^{-1}z' \to \frac{1}{3}(\lambda-2)$, a fact which will be used later.

We consider next the properties of solutions which are *not* defined for all large values of x.

Theorem 4. *If a solution $y = y(x)$ of (1) is defined only in a finite interval $0 \leqslant x < x^*$ then y, y', and y'' are unbounded for $x \to x^*$.*

Proof. If y'' were bounded for $x \to x^*$ then y' and y would also be bounded, whereas $\|\mathbf{y}(x)\| \to \infty$ as $x \to x^*$. If y' were bounded for $x \to x^*$ then y would also be bounded. But then, by (9), y'' would be bounded, and we have seen that this is impossible.

Finally, suppose y were bounded for $x \to x^*$. From (9) we get by integration

$$y'(x) + \tfrac{1}{2}y^2(x) + \tfrac{1}{2}\lambda x^2 + Ax + B = (\lambda+1)\int_0^x \int_0^{\xi} y'^2\,\mathrm{d}\xi\,\mathrm{d}x,$$

where A and B are constants. Since y' is unbounded for $x \to x^*$ the right-hand side is also and $\lambda \neq -1$. But the right-hand side is a monotonic function of x. Therefore it tends to infinity, and so does $y'(x)$. If we put

$$w = \int_0^x \int_0^{\xi} y'^2\,\mathrm{d}\xi\,\mathrm{d}x$$

and $v = w'$ then

$$v\,\mathrm{d}v/\mathrm{d}w = w'' \sim (\lambda+1)^2 w^2 \quad \text{for} \quad w \to \infty.$$

Therefore

$$v^2 \sim \tfrac{2}{3}(\lambda+1)^2 w^3$$

and

$$w' \sim (\tfrac{2}{3})^{\frac{1}{2}}\,|\lambda+1|\,w^{\frac{3}{2}}.$$

By the theory of indeterminate forms it follows that

$$w^{-\frac{1}{2}} \sim c_1(x^* - x) \quad \text{for} \quad x \to x^*,$$

where c_1 is a positive constant. Hence

$$y' \sim (\lambda+1)\,w \sim c_2(x^* - x)^{-2},$$

where $c_2 \neq 0$. Integrating, we get $y \sim c_2(x^* - x)^{-1}$. But this contradicts the original assumption that y is bounded for $x \to x^*$.

Finally we prove

Theorem 5. *Let $y = y(x)$ be a solution of (1). If $\lambda > 0$ then y'^2 is less than 1 at a local maximum of y' and greater than 1 at a local minimum. If $\lambda < 0$ the position is reversed. If $\lambda = 0$ then y' has no local maxima or minima.*

Proof. From the differential equation (1) $y''' = \lambda(y'^2 - 1)$ when $y'' = 0$. At a local maximum we have $y'' = 0$, $y''' \leqslant 0$. If $y''' = 0$ then y'' vanishes identically and there is no strict maximum. Therefore $y''' < 0$ and $y'^2 \lessgtr 1$ according as $\lambda \gtrless 0$. Similarly for a local minimum.

6. Qualitative behaviour of the solutions

In the qualitative theory of differential equations, inaugurated by Poincaré, the main object is to classify the solutions according to their limit-sets. More or less complete results were obtained by Poincaré himself and Bendixson in the case of two-dimensional autonomous systems. For three-dimensional autonomous systems, however, the possibilities are vastly more complicated and, in spite of the efforts of Birkhoff, there is no theory having the same measure of completeness. It is of interest, therefore, to discuss special equations of physical significance from this point of view. In the present section this will be done for the boundary-layer equation.

It will be assumed throughout the discussion which follows that λ *is positive and not equal to* $\frac{1}{2}$. We divide up the entire three-dimensional phase space into 24 domains, as shown in table 1.

Table 1

	$y_1 > 0$		$y_1 < 0$
[1]	$y_2 > 1,\ y_3 > 0,\ y_3' > 0$	[13]	$y_2 > 1,\ y_3 < 0,\ y_3' > 0$
[2]	$y_2 > 1,\ y_3 > 0,\ y_3' < 0$	[14]	$y_2 > 1,\ y_3 < 0,\ y_3' < 0$
[3]	$y_2 > 1,\ y_3 < 0,\ (y_3' > 0)$	[15]	$y_2 > 1,\ y_3 > 0,\ (y_3' > 0)$
[4]	$0 < y_2 < 1,\ y_3 > 0,\ (y_3' < 0)$	[16]	$0 < y_2 < 1,\ y_3 < 0,\ (y_3' < 0)$
[5]	$0 < y_2 < 1,\ y_3 < 0,\ y_3' > 0$	[17]	$0 < y_2 < 1,\ y_3 > 0,\ y_3' > 0$
[6]	$0 < y_2 < 1,\ y_3 < 0,\ y_3' < 0$	[18]	$0 < y_2 < 1,\ y_3 > 0,\ y_3' < 0$
[7]	$-1 < y_2 < 0,\ y_3 > 0,\ (y_3' < 0)$	[19]	$-1 < y_2 < 0,\ y_3 < 0,\ (y_3' < 0)$
[8]	$-1 < y_2 < 0,\ y_3 < 0,\ y_3' > 0$	[20]	$-1 < y_2 < 0,\ y_3 > 0,\ y_3' > 0$
[9]	$-1 < y_2 < 0,\ y_3 < 0,\ y_3' < 0$	[21]	$-1 < y_2 < 0,\ y_3 > 0,\ y_3' < 0$
[10]	$y_2 < -1,\ y_3 > 0,\ y_3' > 0$	[22]	$y_2 < -1,\ y_3 < 0,\ y_3' > 0$
[11]	$y_2 < -1,\ y_3 > 0,\ y_3' < 0$	[23]	$y_2 < -1,\ y_3 < 0,\ y_3' < 0$
[12]	$y_2 < -1,\ y_3 < 0,\ (y_3' > 0)$	[24]	$y_2 < -1,\ y_3 > 0,\ (y_3' > 0)$

In this table the expression $\lambda(y_2^2 - 1) - y_1 y_3$ has been denoted for short by y_3'. The inequality $y_3' \gtrless 0$ is enclosed in brackets when it is a consequence of the other inequalities. A path through a point not belonging to any of these domains passes immediately into one of the domains, unless it is of the type $y_2 \equiv \pm 1$, $y_3 \equiv 0$.†

A domain will be called *transitional* if every path which passes through a point of the domain leaves it at some later date. Otherwise the domain will be called *terminal*. Thus a terminal domain is a domain which contains at least one complete positive half-path. Our first object will be to determine which domains are terminal and which transitional.

If a path tends to infinity for a finite value of x in one of the above domains then, by theorem 4, y_1, y_2, and y_3 all tend to infinity. Moreover, they must all tend to $+\infty$ or all tend to $-\infty$. The only domains in which this is possible are [1] and [23].

If a path remains in one of the domains for all sufficiently large x then, by theorem 2, y_2 tends to ± 1 or $\pm\infty$ and $y_3 \to 0$ or $\pm\infty$ as $x \to \infty$. If $y_2 \to \pm 1$ then $y_3 \to 0$, and if $y_3 \to \pm\infty$

† It is easily seen by differentiation that if $y_2 = y_3' = 0$ then $y_3''' = (2\lambda - 1)\, y_3^2 \neq 0$.

then $y_2 \to \pm\infty$. Moreover, $y_1 \to \pm\infty$ according as y_2 tends to a positive or negative limit. It is easily verified that the only domains in which these conditions are compatible are

[1], [2], [3], [4], [22], [23].

Therefore, these are the only *possible* terminal domains. The corresponding limit-sets are given in table 2.

TABLE 2

domain	limit-set		
[1]	$y_1 \to \infty$,	$y_2 \to \infty$,	$y_3 \to \infty$
[2]	$y_1 \to \infty$,	$y_2 \to \infty$,	$y_3 \to 0$
[3]	$y_1 \to \infty$,	$y_2 \to 1$,	$y_3 \to 0$
[4]	$y_1 \to \infty$,	$y_2 \to 1$,	$y_3 \to 0$
[22]	$y_1 \to -\infty$,	$y_2 \to -\infty$,	$y_3 \to 0$
[23]	$y_1 \to -\infty$,	$y_2 \to -\infty$,	$y_3 \to -\infty$

These possibilities can be further reduced by distinguishing between the two cases $\lambda < \frac{1}{2}$ and $\lambda > \frac{1}{2}$. If we differentiate the last equation (3) we get

$$y_3'' = (2\lambda - 1)\, y_2 y_3 - y_1 y_3'.$$

Integrating again we get

$$y_3'(x) - y_3'(x_0) = (\lambda - \tfrac{1}{2})\,[y_2^2(x) - y_2^2(x_0)] - \int_{x_0}^{x} y_1 y_3' \,\mathrm{d}x. \tag{10}$$

Suppose first that $\lambda < \frac{1}{2}$. Then it follows from (10) that in the domain [1] y_3' is a positive, decreasing function of x. Hence no path can tend to infinity in [1] for a finite value of x. If a path remains in [1] for all $x > x_0$ then $y_2 \to \infty$ as $x \to \infty$. But by (10) this implies that $y_3' \to -\infty$, which is impossible in [1]. Therefore no path can remain in [1] for all sufficiently large x. Thus [1] *is a transitional domain if* $\lambda < \frac{1}{2}$.

Suppose on the other hand that $\lambda > \frac{1}{2}$. In this case we can prove that if $y_2 \to \pm\infty$ as $x \to \infty$ then y_3' cannot have the opposite sign to y_1 for all sufficiently large x. In fact if $y_1 y_3' < 0$ for $x > x_0$ and $y_2 \to \pm\infty$ as $x \to \infty$ then by (10) $y_3' \to +\infty$ as $x \to \infty$. Therefore y_3, y_2, and y_1 all tend to $+\infty$. But this contradicts the hypothesis that $y_1 y_3'$ is negative for x sufficiently large. It follows that [2] *and* [22] *are transitional domains if* $\lambda > \frac{1}{2}$.

We will now show that the half-dozen domains which have not yet been proved to be transitional are in fact terminal. We use for this purpose the following *laws of passage* which govern transition from one domain to another:

General laws of passage

[1] → [2] if $\lambda < \frac{1}{2}$
[2] → [1] if $\lambda > \frac{1}{2}$
[3] → [1] and [5]
[4] → [2] and [6]
[5] → [8] and, if $\lambda > \frac{1}{2}$, [6]
[6] → [9] and, if $\lambda < \frac{1}{2}$, [5]
[7] → [4], [9], and [21]
[8] → [12] and, if $\lambda < \frac{1}{2}$, [9]
[9] → [19] and, if $\lambda > \frac{1}{2}$, [8]
[10] → [24] and, if $\lambda > \frac{1}{2}$, [11]
[11] → [7] and, if $\lambda < \frac{1}{2}$, [10]
[12] → [10] and [22]
[13] → [3], [15] and, if $\lambda > \frac{1}{2}$, [14]
[14] → [16] and, if $\lambda < \frac{1}{2}$, [13]
[15] → [1]
[16] → [6] and [19]
[17] → [15] and, if $\lambda < \frac{1}{2}$, [18]
[18] → [4], [16] and, if $\lambda > \frac{1}{2}$, [17]
[19] → [23]
[20] → [17] and, if $\lambda > \frac{1}{2}$, [21]
[21] → [18], [19] and, if $\lambda < \frac{1}{2}$, [20]
[22] → [24] and, if $\lambda < \frac{1}{2}$, [23]
[23] → [22] if $\lambda > \frac{1}{2}$
[24] → [20]

Special laws of passage

[7] $\rightarrow$ [19]	[12] $\rightarrow$ [24]
[8] $\rightarrow$ [23] if $\lambda < \frac{1}{2}$	[13] $\rightarrow$ [1] and, if $\lambda > \frac{1}{2}$, [6]
[9] $\rightarrow$ [22] if $\lambda > \frac{1}{2}$	[14] $\rightarrow$ [5] if $\lambda < \frac{1}{2}$
[10] $\rightarrow$ [21] if $\lambda > \frac{1}{2}$	[17] $\rightarrow$ [2] if $\lambda < \frac{1}{2}$
[11] $\rightarrow$ [20] if $\lambda < \frac{1}{2}$	[18] $\rightarrow$ [6] and, if $\lambda > \frac{1}{2}$, [1]

The general laws of passage give the permissible transitions across a bounding face. The special laws of passage give the permissible transitions across a bounding edge, if they are not already included in the general laws. The proofs of these laws are extremely simple and will be sufficiently illustrated by one example.

Consider the domain [5]. In this domain y_1 and y_3 are increasing functions and y_2 is a decreasing function. Hence the only faces through which a path can possibly leave the domain are the faces $y_2 = 0$, $y_3 = 0$, and $y_3' = 0$. If it leaves through the face $y_3 = 0$ then at the point of departure $y_3' = \lambda(y_2^2 - 1) \leqslant 0$. Since y_3' is positive in [5] this is only possible if $y_3' = 0$. But y_3 and y_3' cannot vanish simultaneously. Therefore the face $y_3 = 0$ is ruled out. If a path leaves through the face $y_3' = 0$, then at the point of departure $y_3'' = (2\lambda - 1)\, y_2 y_3 \leqslant 0$. This is only possible if $y_2 = 0$ or $\lambda > \frac{1}{2}$. A path which leaves through the face $y_3' = 0$ enters the domain [6]. A path which leaves through the face $y_2 = 0$ or the edge $y_2 = y_3' = 0$ enters the domain [8]. Moreover, these possibilities all actually occur.

It can be seen immediately from the table that no path leaves either [2] or [23] if $\lambda < \frac{1}{2}$, and that no path leaves [1] if $\lambda > \frac{1}{2}$. Hence these domains are certainly terminal for the values of λ stated. It has already been proved in § 2 that the domains [3] and [4] are terminal for all $\lambda > 0$. In the same way it can be shown that [22] is terminal if $\lambda < \frac{1}{2}$. The points of [22] belonging to paths which pass from [22] to [24] form a non-empty open subset. The points belonging to paths which pass from [22] to [23] form another open subset, which is non-empty if $\lambda < \frac{1}{2}$. Since [22] is connected it follows that it is a terminal domain for $\lambda < \frac{1}{2}$. The only domain whose character still remains in doubt is [23] for $\lambda > \frac{1}{2}$. We will return to this point in § 7. Anticipating the results established there we can state finally:

The only terminal domains are

[3] *and* [4] *for all* $\lambda > 0$, [2] *and* [22] *for* $\lambda < \frac{1}{2}$, [1] *for* $\lambda > \frac{1}{2}$ *and* [23] *for* $\lambda < 2$.

We have also to consider the possibility of paths which do not terminate in any domain, but wander for ever from one domain to another. A solution will evidently be of this kind if y'' vanishes infinitely often. We are going to show that the converse is also true. Suppose in fact that y'' vanishes at most a finite number of times. Then there exists a value x_0 such that $y'' \neq 0$ for all $x > x_0$ for which y is defined. Therefore y' and y are ultimately of constant sign. But $y^{\text{iv}} = (2\lambda - 1)\, y' y''$ when $y''' = 0$. It follows that y''' is ultimately of constant sign, since y^{iv} must have opposite signs at consecutive zeros of y'''. Hence the corresponding path terminates in one of the 24 domains.

A solution for which y'' vanishes infinitely often without vanishing identically will be called *oscillatory*. It is possible that oscillatory solutions exist for all values of λ greater than 1. However, it will now be shown that they do not exist if $\lambda \leqslant 1$. We again consider separately the cases $\lambda < \frac{1}{2}$ and $\lambda > \frac{1}{2}$. For $\lambda < \frac{1}{2}$ considerably more can be said, and we consider this case first.

Lemma 1. *If* $\lambda < \frac{1}{2}$ *then every path which enters* [20] *from* [24] *leaves* [17] *by* [15], *and every path which enters* [5] *from* [6] *leaves* [8] *by* [9].

The proof is based on (10). Suppose a path entered [20] from [24] for $x = x_0$, passed into [17] and left again by [18] for $x = x_1$. Then $y_2^2(x_0) = 1$, $y_1 y_3' < 0$ for $x_0 < x < x_1$, and $y_2^2(x_1) \leqslant 1$. Therefore the right side of (10) is positive for $x = x_1$. But the left side of (10) is negative for $x = x_1$, since $y_3'(x_0) > 0$ and $y_3'(x_1) = 0$. Thus we have a contradiction. The second part of the lemma is proved in the same way.

Theorem 6. *Let* $\lambda < \frac{1}{2}$. *If* $y'^2 > 1$, $y'' \geqslant 0$, *and* $y''' > 0$ *at* $x = 0$ *then* y *is defined and* y'' *is positive for all* $x > 0$. *Moreover, the corresponding path terminates in* [2].

Proof. For small positive values of x the path lies in one of the domains [10], [24], [15], [1]. Its subsequent travels are uniquely determined by the laws of passage and the first part of the preceding lemma:

$$[10] \to [24] \to [20] \to [17] \to [15] \to [1] \to [2].$$

It follows that y is defined for all $x > 0$, since no path can tend to infinity in [2] for a finite value of x. Furthermore, $y_3 > 0$ in all the domains passed through.

Lemma 2. *If* $\lambda < \frac{1}{2}$ *then every path which passes through a point of* [23] *tends to infinity for a finite value of* x.

To prove this we use the differential equation

$$y^{\text{iv}} = (2\lambda - 1)\, y' y'' - y y''', \tag{11}$$

which is obtained from (1) by differentiation. For $\lambda < \frac{1}{2}$ the right-hand side is a non-decreasing function of y, y', and y'' in [23]. Moreover, the equation has the elementary solution

$$y = \alpha/(x_0 - x),$$

where $\alpha = 6/(\lambda - 2)$ and x_0 is an arbitrary constant. Let $y(x)$ be any solution which passes through a point of [23] for $x = 0$, and therefore remains in [23] for all positive values of x for which it is defined. We can choose $x_0 > 0$ so large that

$$\alpha x_0^{-1} \geqslant y(0), \quad \alpha x_0^{-2} \geqslant y'(0), \quad 2\alpha x_0^{-3} \geqslant y''(0), \quad 6\alpha x_0^{-4} \geqslant y'''(0).$$

Then, by Kamke's theorem, $$\frac{\alpha}{x_0 - x} \geqslant y(x) \quad (x > 0).$$

It follows that $y(x)$ cannot be defined for all positive values of x, since the left side of this inequality tends to $-\infty$ as $x \to x_0 - 0$.

Theorem 7. *Let* $\lambda < \frac{1}{2}$. *If* $y'^2 < 1$, $y'' \leqslant 0$, *and* $y''' < 0$ *at* $x = 0$ *then* y *is defined only in a finite interval* $0 < x < x^*$ *and* y'' *is negative throughout this interval. Moreover, the corresponding path terminates in* [23].

Proof. For small positive values of x the path lies in one of the domains [16], [6], [9], [19]. Its subsequent travels are determined by the laws of passage and the second part of lemma 1:

$$[16] \to [6] \to [5] \to [8] \to [9] \to [19] \to [23].$$

Therefore, by lemma 2, it is not defined for all $x > 0$. In all the domains passed through $y_3 < 0$.

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THEOREM 8. *Let $\lambda < \frac{1}{2}$. Then the second derivative y'' of any solution of the boundary-layer equation vanishes either identically or at most once in its entire interval of definition.*

Proof. If y'' does not vanish identically then its zeros are simple. Suppose $y''(\xi) = 0$ and $y'''(\xi) > 0$. Then $y'^2(\xi) > 1$, by theorem 5. Therefore $y''(x) > 0$ for all $x > \xi$, by theorem 6. It follows from theorem 1 that we must also have $y''(x) < 0$ for all $x < \xi$.

Again, suppose $y''(\xi) = 0$ and $y'''(\xi) < 0$. Then $y'^2(\xi) < 1$. Therefore, by theorem 7, $y''(x) < 0$ for all $x > \xi$ for which y is defined. By theorem 1 it further follows that $y''(x) > 0$ for all $x < \xi$ for which y is defined.

We have actually proved rather more than is stated in theorem 8. In fact the proof shows that if y' has a minimum value then y is defined for $-\infty < x < \infty$, and the corresponding path is 'born' in [14] and 'dies' in [2]. If y' has a maximum value then y is defined only in a finite interval $x_1 < x < x_2$, and the corresponding path is born in [11] and dies in [23].

An immediate corollary of theorem 8 is the non-existence of oscillatory solutions for $\lambda < \frac{1}{2}$. It will be shown next that the same holds if $\frac{1}{2} < \lambda \leqslant 1$, although the proof in this case is necessarily very different.

LEMMA 3. *If $\frac{1}{2} < \lambda \leqslant 1$ then every path which enters* [22] *from* [23], *and likewise every path which enters* [8] *or* [22] *from* [9], *leaves* [20] *by* [17].

Proof. From (11) we get by differentiation

$$y^{\mathrm{v}} + yy^{\mathrm{iv}} = (2\lambda - 1)\, y''^2 + 2(\lambda - 1)\, y'y'''.$$

The right-hand side is non-negative if $y'y''' \leqslant 0$. Therefore $w = y^{\mathrm{iv}}$ satisfies the differential inequality

$$w' \geqslant -yw$$

in a region where $y'y''' \leqslant 0$. It follows that if $w(x_0) \geqslant 0$ then $w(x) \geqslant 0$ for $x \geqslant x_0$. Moreover, $w(x_1) = 0$ implies $w(x) \equiv 0$ for $x_0 \leqslant x \leqslant x_1$. Thus if $y^{\mathrm{iv}}(x_0) \geqslant 0$ and $y'y''' \leqslant 0$ for $x_0 \leqslant x \leqslant x_1$ then $y^{\mathrm{iv}}(x_1) > 0$, unless y'' vanishes identically.

Consider now a path which enters [22] from [23] for $x = x_0$. By the laws of passage it must pass into [24] and then into [20]. Suppose it leaves [20] for $x = x_1$. Then

$$y^{\mathrm{iv}} = (2\lambda - 1)\, y'y'' > 0 \quad \text{for} \quad x = x_0 \qquad \text{and} \qquad y'y''' < 0 \quad \text{for} \quad x_0 < x < x_1.$$

Hence $y^{\mathrm{iv}} > 0$ for $x = x_1$. But if the path left [20] by [21] we would have

$$y^{\mathrm{iv}} = (2\lambda - 1)\, y'y'' < 0 \quad \text{for} \quad x = x_1.$$

Therefore it leaves via [17].

The other part of the lemma is proved in the same way, although the sequence of domains through which the path passes is not uniquely determined. The paths referred to in lemma 3 all terminate in [1]. For by the laws of passage every path which enters [17] passes into [15] and then terminates in [1].

Suppose now that there is a path on which y'' vanishes infinitely often, without vanishing identically. Since $y'^2 < 1$ at a local maximum of y' and $y'^2 > 1$ at a local minimum we must have $y' < -1$ at a minimum which follows a maximum. Hence it happens infinitely often that

$$y' < -1, \quad y'' = 0, \quad y''' > 0.$$

Thus the path passes infinitely often from [12] to [10] or from [22] to [24]. But by the preceding lemma and the remark which follows it the path cannot enter [22] from [23] or [9]. It must therefore enter it from [12]. The only domain from which it can enter [12] is [8]. By lemma 3 again it cannot enter [8] from [9]. It must therefore enter it from [5]. But [5] can only be entered from [3], [3] can only be entered from [13], and [13] cannot be entered at all. Thus we have arrived at a contradiction.†

Altogether we have proved

THEOREM 9. *If* $\lambda \leqslant 1$ *every path terminates in one of the* 24 *domains, unless it is of the form* $y \equiv x_0 \pm x$.

7. ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS

Our next task will be to determine the asymptotic form of the solutions considered in the last section. This has already been done in § 4 for the paths which terminate in the domains [3] and [4]. We will now discuss in turn the paths which terminate in [2], [22], [1], and [23].

(i) *Paths terminating in* [2]: $\lambda < \frac{1}{2}$

The differential equation (4) can be written in the form

$$zz''+z'^2+yz' = \lambda(z^2-1)/z. \qquad (12)‡$$

But on any path in [2] $y' = z > 1$ and $y''' = z(zz''+z'^2) < 0$. Hence

$$\frac{zz'}{z^2-1} > \frac{\lambda}{y}.$$

This inequality states that the logarithmic derivative of $(z^2-1)/y^{2\lambda}$ is positive. Therefore $(z^2-1)/y^{2\lambda}$ is an increasing function of y. In particular

$$z^2 > 1+cy^{2\lambda}, \qquad (13)$$

where c is a positive constant.

Again, if we multiply (12) by $y^{-1-\lambda}$ we get

$$\frac{d}{dy}(y^{-\lambda}z)+\frac{1}{y^{1+\lambda}}\frac{d}{dy}(zz')+\frac{\lambda}{y^{1+\lambda}z} = 0.$$

Integration with respect to y gives

$$y^{-\lambda}z+\frac{zz'}{y^{1+\lambda}}+(1+\lambda)\int_{y_0}^{y}\frac{zz'}{y^{2+\lambda}}dy+\lambda\int_{y_0}^{y}\frac{dy}{y^{1+\lambda}z} = \text{constant}.$$

Letting $y\to\infty$ we see that $y^{-\lambda}z$ tends to a finite limit, since $y' = z\to\infty$ and $y'' = zz'\to 0$ in [2]. Moreover, by (13) this limit must be positive. Thus

THEOREM 10. *On every path which terminates in* [2]

$$z \sim ay^{\lambda} \quad \textit{for} \quad y\to\infty \quad (a>0).$$

To obtain a closer approximation we rewrite the last equation in the form

$$y^{-\lambda}z+\frac{zz'}{y^{1+\lambda}}-(1+\lambda)\int_{y}^{\infty}\frac{zz'}{y^{2+\lambda}}dy-\lambda\int_{y}^{\infty}\frac{dy}{y^{1+\lambda}z} = a. \qquad (14)$$

† A more detailed analysis shows that y'' vanishes at most three times, if it does not vanish identically.
‡ No confusion will be caused by using dashes also to denote differentiation with respect to y.

The second and third terms on the left are of order $o(y^{-1-\lambda})$. By what we have just proved the fourth term is equal to $[-1/2a+o(1)]\,y^{-2\lambda}$. Since $\lambda < \frac{1}{2}$ the second and third terms are negligible in comparison with this and we obtain

$$z \approx ay^{\lambda} + \frac{1}{2ay^{\lambda}}.$$

This could also be written $z \doteqdot (1+a^2y^{2\lambda})^{\frac{1}{2}}$.

The asymptotic behaviour of y as a function of x can be obtained by integration: $y \sim bx^{1/(1-\lambda)}$ for $x \to \infty$, where $b > 0$. However, the relationship between y' and y appears to be more fundamental than that between y and x.

(ii) *Paths terminating in* [22]: $\lambda < \frac{1}{2}$

In [22] we have $y < 0$, $z < -1$, $z' > 0$, and $zz''+z'^2 < 0$. Therefore, by (12), $yz' > \lambda z$. Hence $|y|^{-\lambda} z$ increases when y decreases and tends to a finite limit $a \leqslant 0$ as $y \to -\infty$. We will show that $a \neq 0$.

On any path which remains in [22] $y \to -\infty$, $z \to -\infty$, and $zz' \to 0$. Multiplying (12) by $|y|^{-1-\lambda}$ and integrating with respect to y we get

$$\frac{z}{|y|^{\lambda}} - \frac{zz'}{|y|^{1+\lambda}} + (1+\lambda)\int_{-\infty}^{y} \frac{zz'}{|y|^{2+\lambda}}\,\mathrm{d}y - \lambda\int_{-\infty}^{y}\frac{\mathrm{d}y}{|y|^{1+\lambda}z} = a.$$

The second and third terms on the left are $o(|y|^{-1-\lambda})$. The fourth term can be estimated as follows:

$$0 < -\lambda\int_{-\infty}^{y}\frac{\mathrm{d}y}{|y|^{1+\lambda}z} < -\frac{\lambda}{z}\int_{-\infty}^{y}\frac{\mathrm{d}y}{|y|^{1+\lambda}} = -\frac{1}{|y|^{\lambda}z}.$$

Consequently $a = 0$ implies $z - \theta z^{-1} \to 0$, where $0 < \theta < 1$. But this is impossible because $z \to -\infty$. Hence

THEOREM 11. *On every path which terminates in* [22]

$$z \sim a\,|y|^{\lambda} \quad \textit{for} \quad y \to -\infty \quad (a < 0).$$

In the same way as before we can obtain the improved approximation

$$z \approx a\,|y|^{\lambda} + \frac{1}{2a\,|y|^{\lambda}}.$$

The asymptotic behaviour of y as a function of x is given by $y \sim bx^{1/(1-\lambda)}$ for $x \to \infty$, where $b < 0$.

It will now be shown that *if* $\lambda < 2$, *then for every constant* $a < 0$ *the equation* (12) *has a unique solution for which* $z \sim a\,|y|^{\lambda}$ *as* $y \to -\infty$. The change of variables

$$w = (z^2-1)/|y|^{2\lambda}, \quad |y| = \mathrm{e}^t$$

transforms the equation (12) into

$$\ddot{w} + (4\lambda - 1 + \mathrm{e}^{2t}/z)\,\dot{w} + 2\lambda(2\lambda-1)\,w = 0.$$

We will write this in the form

$$\ddot{w} + p(t)\,\dot{w} = q(t), \tag{*}$$

where
$$p(t) = 4\lambda - 1 + \mathrm{e}^{2t}/z, \quad q(t) = 2\lambda(1-2\lambda)\,w.$$

If $z \sim a|y|^\lambda$ then $w \to a^2$ and $p(t) \sim a^{-1} e^{(2-\lambda)t}$. Thus $p(t)$ tends rapidly to $-\infty$. Solving the first-order linear equation (*) for $\dot{w}$ we get

$$\dot{w} \exp\left(\int_{t_0}^{t} p(\tau)\, d\tau\right) = c - \int_t^\infty q(s) \exp\left(\int_{t_0}^{s} p(\tau)\, d\tau\right) ds,$$

where c is a constant. The right-hand side tends to c as $t \to \infty$. Since $\dot{w}$ cannot tend to $\pm\infty$ we must therefore have $c = 0$. Thus

$$\dot{w} = -\int_t^\infty q(s) \exp\left(\int_t^s p(\tau)\, d\tau\right) ds$$

and $\dot{w}$ tends to zero as $t \to \infty$. It may be shown by the method of successive approximations (cf. § 9) that there is one and only one continuously differentiable function $w = w(t)$ defined for all sufficiently large t which satisfies this integro-differential equation and tends to a^2 as $t \to \infty$.

Since
$$2zz' = -|y|^{2\lambda-1}(\dot{w} + 2\lambda w)$$
it follows that
$$y'' = zz' \sim -\lambda a^2 |y|^{2\lambda-1}.$$

Thus y'' tends to 0 if $\lambda < \frac{1}{2}$ and to $-\infty$ if $\lambda > \frac{1}{2}$. Therefore the solution whose existence we have proved terminates in [22] or [23] according as λ is less than or greater than $\frac{1}{2}$.

Similarly, it may be shown that *if* $\lambda < 2$, *then for every constant* $a > 0$ *the equation* (12) *has at least one solution for which* $z \sim ay^\lambda$ *as* $y \to \infty$. However, the solution is no longer unique. In fact, since $p(t) \to +\infty$, the corresponding constant c may now be different from zero. The solutions terminate in [2] or [1] according as λ is less than or greater than $\frac{1}{2}$.

(iii) *Paths terminating in* [1]: $\lambda > \frac{1}{2}$

It follows from (12) that on any path in [1]

$$yz' < \lambda z. \tag{15}$$

Hence $y^{-\lambda}z$ decreases as y increases and tends to a finite non-negative limit a as $y \to \infty$. We will show that this limit cannot be zero *if* $\lambda < 2$.

Suppose on the contrary that $z = o(y^\lambda)$. Moreover, let μ be any positive number not exceeding λ such that $z = o(y^\mu)$. Then $z' = o(y^{\mu-1})$ by (15). Therefore the equation (14) has the form

$$y^{-\lambda}z + o(y^{2\mu-2-\lambda}) - \theta y^{-\lambda} z^{-1} = 0,$$

where $0 < \theta < 1$, or

$$z - \theta z^{-1} = o(y^{2\mu-2}).$$

If $\mu \leqslant 1$ this immediately gives a contradiction, since $z \to \infty$. On the other hand, if $\mu > 1$ we obtain $z = o(y^{2\mu-2})$. Thus from $z = o(y^{\mu_0})$ we can deduce $z = o(y^{\mu_1})$, where $\mu_1 = 2\mu_0 - 2$. But the solution of the difference equation $\mu_{n+1} = 2\mu_n - 2$ is

$$\mu_n = 2 - (2 - \mu_0)\, 2^n.$$

Therefore a finite number of repetitions of this process will produce a positive exponent $\mu_n \leqslant 1$ and thus a contradiction. Hence

THEOREM 12. *If* $\lambda < 2$ *then on every path which terminates in* [1]

$$z \sim ay^\lambda \quad \textit{for} \quad y \to \infty \quad (a > 0).$$

We show next that this asymptotic expression for z can be differentiated. From the inequalities $zz'' + z'^2 > 0$ and $yz' < \lambda z$ it follows that

$$z'' > -\lambda^2 z/y^2 = O(y^{\lambda-2}).$$

By a theorem due to Hardy, Littlewood and Landau (see Widder 1946) we can infer from this that $z' \sim \lambda a y^{\lambda-1}$.

Improved approximations for z can now be deduced by substitution in (14). We have

$$y^{-\lambda} z + \lambda a^2 y^{\lambda-2} - \frac{\lambda(1+\lambda)}{2-\lambda} a^2 y^{\lambda-2} - \frac{1}{2a} y^{-2\lambda} \approx a.$$

Hence the precise asymptotic form depends on whether $\lambda - 2 \gtrless -2\lambda$:

$$z \approx \begin{cases} ay^{\lambda} + \dfrac{1}{2ay^{\lambda}} & \text{if } \tfrac{1}{2} < \lambda < \tfrac{2}{3}, \\ ay^{\frac{2}{3}} + \dfrac{a^3+3}{6ay^{\frac{2}{3}}} & \text{if } \lambda = \tfrac{2}{3}, \\ ay^{\lambda} + \dfrac{\lambda(2\lambda-1)}{2-\lambda} \dfrac{a^2}{y^{2-2\lambda}} & \text{if } \tfrac{2}{3} < \lambda < 2. \end{cases}$$

The asymptotic behaviour of y as a function of x introduces a new feature in that the solutions may now become infinite for a finite value of x. It is easily shown that

$$y \sim bx^{1/(1-\lambda)} \quad \text{for} \quad x \to \infty \quad \text{if} \quad \tfrac{1}{2} < \lambda < 1,$$

$$y \sim b(x^* - x)^{1/(1-\lambda)} \quad \text{for} \quad x \to x^* - 0 \quad \text{if} \quad 1 < \lambda < 2,$$

where b is a positive constant.

The behaviour of the solutions is quite different when $\lambda > 2$. This is shown by the following

THEOREM 13. *If* $\lambda > 2$ *then on every path which terminates in* [1]

$$z \sim \rho y^2 \quad \textit{for} \quad y \to \infty,$$

where $\rho = \frac{1}{6}(\lambda - 2)$.

The change of variables $z = y^2 u$, $|y| = e^t$ replaces the differential equation (12) by the equation

$$\ddot{u} + 7\dot{u} + 6u + \dot{u}^2/u + \dot{u}/u + 2 - \lambda = -\lambda/e^{4t}u^2, \tag{16}$$

where dots denote differentiation with respect to t. In the new variables the inequalities which define the domain [1] become

$$u > e^{-2t}, \quad \dot{u} > -2u, \quad \dot{u} < (\lambda - 2)\, u - \lambda/e^{4t} u. \tag{17}$$

Moreover, on any path which remains in [1] $e^{2t} u$ tends to infinity with t.

The proof of the theorem will be conducted in a number of stages:

1. *If u tends to a finite limit as $t \to \infty$ this limit must be ρ and $\dot{u}$ must tend to zero.*

Suppose first that $u \to 0$. Then by (17) also $\dot{u} \to 0$. It follows from the differential equation (16) that

$$\ddot{u} + \dot{u}^2/u + \dot{u}/u \to \lambda - 2,$$

or

$$\frac{\ddot{u}}{1+\dot{u}} + \frac{\dot{u}}{u} \to \lambda - 2.$$

Integrating, we get

$$\ln(1+\dot{u}) + \ln u \sim (\lambda - 2)\, t.$$

Hence $\ln u \sim (\lambda - 2)\, t$, which contradicts the original supposition.

Thus $u \to 0$ is impossible. Therefore, by theorem 3, $u \to \rho$. Moreover, by the remark following the proof of theorem 3, $\dot{u} + 2u \to 2\rho$. Therefore $\dot{u} \to 0$.

2. *If $u \neq \rho$ for all sufficiently large t then u tends to a finite limit as $t \to \infty$.*

Let
$$V = \tfrac{1}{12}(7\lambda - 8)\, u^2 + \rho u \dot{u}. \tag{18}$$

It is easily verified, using the differential equation (16), that
$$\dot{V} = (\rho - u)\,[(\lambda - 2)\, u - \dot{u}] - \lambda\rho/e^{4t} u.$$

If $u \leqslant \rho$ for all sufficiently large t then, by (17),
$$\dot{V} \geqslant (\rho - u)\,\lambda/e^{4t} u - \lambda\rho/e^{4t} u = -\lambda\, e^{-4t}.$$

Therefore
$$W = V - \tfrac{1}{4}\lambda\, e^{-4t}$$
is an increasing function of t and tends to a limit l, finite or infinite, as $t \to \infty$. Hence also $V \to l$, since the exponential function tends to zero. But this implies that $v = u^2$ has a definite limit. For, in general, if $v + a^{-1}\dot{v} \to l$, where a is a positive constant, then $v \to l$.

{*Proof*. Let $f = v\, e^{at}$ and $g = e^{at}$. Then by the theory of indeterminate forms
$$\lim v = \lim f/g = \lim \dot{f}/\dot{g} = \lim\, (v + a^{-1}\dot{v}).\}$$

Therefore u has a well-determined limit, which must be finite and not greater than ρ.

If $u \geqslant \rho$ for all sufficiently large t then W is a decreasing function and tends to a limit l $(-\infty \leqslant l < \infty)$. Hence also $V \to l$. As before this implies that
$$u^2 \to \frac{12}{7\lambda - 8}\, l.$$

Therefore l is finite and u tends to a finite limit.

3. *If u has a local minimum at $t = t_0$ and $u < \rho$ for $t = t_0$ then $u < \rho$ for all $t > t_0$.*

We will assume that u takes the value ρ for some value of t greater than t_0 and deduce a contradiction. Let t_2 be the least value of t after t_0 for which $u = \rho$. Then $\dot{u} > 0$ in some interval $t_2 - \delta < t < t_2$. We are given that $\dot{u} = 0$ for $t = t_0$. Let t_1 be the greatest value of t in the interval $t_0 \leqslant t < t_2$ for which $\dot{u} = 0$. Then $u < \rho$ and $\dot{u} > 0$ for $t_1 < t < t_2$.

Now by (16) $\dot{u} \geqslant 0$ implies
$$\ddot{u} + 7\dot{u} + 6u < \lambda - 2.$$

This can be written in the form $\dot{w} + 6w < \lambda - 2$, where $w = u + \dot{u}$. In particular, we must have $\dot{w} < 0$ when $w = \rho$. Consequently, if w is less than ρ initially it remains less than ρ for as long as $\dot{u} \geqslant 0$. But $w < \rho$ for $t = t_1$. Therefore $w < \rho$ for $t = t_2$. This is the required contradiction.

Theorem 13 now follows at once. For by stages 1 and 2 it is sufficient to show that on no path in [1] do we have both $u > \rho$ and $u < \rho$ for arbitrarily large values of t, and this is assured by stage 3.

To obtain a closer approximation to the solutions we proceed as follows. Let $\zeta = u - \rho$. Then for small values of ζ and $\dot{\zeta}$
$$\ddot{\zeta} + (7 + \rho^{-1})\,\dot{\zeta} + 6\zeta \doteqdot -\lambda\rho^{-2}\, e^{-4t}.\dagger$$

The solutions of this second-order linear differential equation are
$$\zeta = c_1\, e^{\mu_1 t} + c_2\, e^{\mu_2 t} + \frac{6\lambda}{\lambda^2 - 4}\, e^{-4t},$$

where c_1, c_2 are arbitrary constants and μ_1, μ_2 are the roots of the quadratic equation
$$\mu^2 + (7 + \rho^{-1})\,\mu + 6 = 0.$$

† This process of linearization can be justified by the method of successive approximations.

μ_1 and μ_2 cannot both be less than -4, since their product is equal to 6. On the other hand, for $\lambda > 2$ the smaller root is less than -4, since the quadratic is negative when $\mu = -4$.

Hence if $\lambda > 2$ the solutions which terminate in [1] have one or other of the forms

$$u-\rho \sim \frac{6\lambda}{\lambda^2-4}\,\mathrm{e}^{-4t},$$

$$u-\rho \sim c\,\mathrm{e}^{\mu t},$$

where
$$\mu = -\tfrac{1}{2}(7+\rho^{-1})+\tfrac{1}{2}[(7+\rho^{-1})^2-24]^{\frac{1}{2}}.$$

In terms of the original variables these results read

$$z \approx \rho y^2 + \frac{6\lambda}{\lambda^2-4}\,y^{-2},$$

$$z \approx \rho y^2 + c y^{2+\mu} \quad (c \neq 0).$$

The asymptotic behaviour of y as a function of x is given by $y \sim \rho^{-1}(x^*-x)^{-1}$.

(iv) *Paths terminating in* [23]

We again make the change of variables $z = y^2 u$, $|y| = \mathrm{e}^t$, but now $t \to \infty$ as $y \to -\infty$. In the new variables the domain [23] is defined by the inequalities

$$u < -\mathrm{e}^{-2t}, \quad \dot{u} < -2u, \quad \dot{u} < (\lambda-2)\,u - \lambda/\mathrm{e}^{4t}u. \tag{19}$$

We show first that u and $\dot{u}$ are bounded on any path which remains in [23].

It follows from (12) that

$$\mathrm{d}(zz'+yz)/\mathrm{d}y = z+\lambda(z^2-1)/z < -1.$$

Hence, by integration, $\quad zz'+yz > c_1 - y \quad$ for $\quad y < y_1$,

where c_1 is a constant. Dividing by z we get

$$z'+y < 0 \quad \text{for} \quad y < y_2 < y_1,$$

and therefore
$$z + \tfrac{1}{2}y^2 > c_2.$$

Thus
$$\underline{\lim}\, y^{-2}z \geqslant -\tfrac{1}{2}, \quad \underline{\lim}\, y^{-1}z' \geqslant -1.$$

Since the corresponding upper limits are certainly not greater than zero $y^{-2}z = u$ and $y^{-1}z' = \dot{u}+2u$ are bounded. Moreover,

$$-\tfrac{1}{2} \leqslant \overline{\lim}\, u \leqslant 0, \quad -1 \leqslant \overline{\lim}\, \dot{u} \leqslant 1.$$

It will be shown next that [23] *is a transitional domain if* $\lambda > 2$. In fact, by (19),

$$\dot{u}/u > \lambda - 2 - \lambda/\mathrm{e}^{4t}u^2$$

and $z = \mathrm{e}^{2t}u \to -\infty$ on any path which remains in [23]. Hence $\dot{u}/u \geqslant \delta > 0$ for all sufficiently large t. It follows on integration that $u \to -\infty$, contrary to what has just been proved. We have already seen in (ii) that [23] *is a terminal domain for* $\frac{1}{2} < \lambda < 2$. The point left open in § 6 has thus been completely settled.

On any path which terminates in [23] $\lambda/\mathrm{e}^{4t}u^2$ tends to zero. It may be expected, therefore, that the corresponding solutions are asymptotically equivalent to solutions of the autonomous equation

$$\ddot{u}+7\dot{u}+6u+\dot{u}^2/u+\dot{u}/u+2-\lambda = 0$$

or, what is the same, to solutions of the autonomous system

$$\dot{u} = v,$$
$$\dot{v} = \lambda - 2 - 6u - 7v - v(1+v)/u.$$

To get rid of the denominator we will change the path parameter from t to

$$\tau = -\int_{t_0}^{t} \frac{\mathrm{d}t}{u}.$$

Then the system just written has the same paths in the half-plane $u < 0$ as the system

$$\left.\begin{aligned}\dot{u} &= -uv,\\ \dot{v} &= (2-\lambda)\,u + v + 6u^2 + 7uv + v^2,\end{aligned}\right\} \tag{20}$$

and they are described in the same sense. (But the dots now denote differentiation with respect to τ.) On any half-path on which u is bounded away from zero $t \to \infty$ is equivalent to $\tau \to \infty$.

The system (20) will be studied in detail in the next section. In particular it will be shown that if $0 < \lambda < \frac{1}{2}$, then every path which passes through a point of the domain $u < 0$, $v < (\lambda - 2)\,u$ converges to the point $(\rho, 0)$ as $t \to \infty$. We are now going to deduce from this:

THEOREM 14. *If $\lambda < \frac{1}{2}$ then on every path which terminates in* [23]

$$z \sim \rho y^2 \quad \textit{for} \quad y \to -\infty.$$

To every solution of (16) corresponds a solution of the system

$$\dot{u} = v,$$
$$\dot{v} = \lambda - 2 - 6u - 7v - v(1+v)/u - \lambda w/u^2,$$
$$\dot{w} = -4w.$$

This last system has the asymptotically stable solution $u \equiv \rho$, $v = w \equiv 0$. In fact all characteristic roots of the variational equation have negative real parts. We can therefore find numbers $\delta > 0$ and T_0 such that $u \to \rho$ and $\dot{u} \to 0$ on every path of the equation (16) which passes through a point of the δ-neighbourhood

$$|u - \rho| \leqslant \delta, \quad |\dot{u}| \leqslant \delta$$

for a value of t greater than T_0.

Let R be any compact subset of the domain $u < 0$, $v < (\lambda - 2)\,u$. Then there exists a value L such that all paths of the autonomous system (20) which pass through a point of R at $t = 0$ lie in the $\frac{1}{2}\delta$-neighbourhood of the point $(\rho, 0)$ for all $t \geqslant L$. Furthermore, since the solutions of the differential equations we are considering depend continuously on the right-hand sides, there exists a value $T_1 = T_1(R)$, which we may suppose greater than T_0, such that every path of the equation (16) which passes through a point of R for a value $t \geqslant T_1$ remains throughout the interval $T_1 \leqslant t \leqslant T_1 + L$ in the $\frac{1}{2}\delta$-neighbourhood of the path of the system (20) which passes through the same point of R at the same time. It follows that $u \to \rho$ and $\dot{u} \to 0$ on every path of the equation (16) which passes through a point of R for a value of t greater than $T_1(R)$.

There remains the possibility that some path lies outside every compact subset of the domain $u < 0$, $v < (\lambda - 2)\,u$ for all sufficiently large t. It is easily seen that this implies that

$u \to 0$ and $\dot{u} \to 0$ when $t \to \infty$. As in (iii), we can deduce from this that $\ln u \sim (\lambda - 2)\,t$. Hence

$$z = -\mathrm{e}^{[\lambda + o(1)]\,t} \quad \text{and} \quad w = (z^2 - 1)/|y|^{2\lambda} = \mathrm{e}^{o(t)}.$$

By the argument which we used in (ii) it follows that $\dot{w} \to 0$ as $t \to \infty$. Moreover, it tends to zero so rapidly that w has a finite limit for $t \to \infty$. This limit is necessarily positive, since w increases with t on any path in [23]. Therefore

$$z \sim a\,|y|^{\lambda} \quad \text{and} \quad zz' \sim -\lambda a^2\,|y|^{2\lambda - 1} \quad \text{for} \quad y \to -\infty \quad (a < 0).$$

But this is impossible for $\lambda \leqslant \frac{1}{2}$, because $zz' \to -\infty$ on any path which terminates in [23]. This completes the proof of theorem 14.

Improved approximations can be obtained in the same way as before. The roots of the quadratic equation

$$\mu^2 + (7 + \rho^{-1})\,\mu + 6 = 0$$

are now conjugate complex numbers with real part greater than -4. Consequently

$$u \approx \rho + A\,\mathrm{e}^{-at} \cos\,(bt + \epsilon),$$

where

$$a = \tfrac{1}{2}(7 + \rho^{-1}), \quad b = (6 - a^2)^{\frac{1}{2}},$$

and A and ϵ are constants.

8. Study of a related system

The present section, which is independent of the rest of the paper, is devoted to a study of the system

$$\left.\begin{aligned} \dot{u} &= -uv, \\ \dot{v} &= (2 - \lambda)\,u + v + 6u^2 + 7uv + v^2, \end{aligned}\right\} \tag{20}$$

in the half-plane $u < 0$ *for* $0 \leqslant \lambda < 2$.

In accordance with general theory we first determine the position and nature of the critical points. The critical points in the finite plane are the intersections of the hyperbola $\dot{v} = 0$ with the co-ordinate axes. There are three points of intersection, namely, the points $(0, -1)$, $(0, 0)$ and $(\rho, 0)$, where $\rho = \frac{1}{6}(\lambda - 2)$. These three critical points will be denoted by A, B, and C, respectively. (See figure 1.)

In the neighbourhood of A we have

$$\dot{u} = u + \ldots,$$
$$\dot{v} = -(\lambda + 5)\,u - (v + 1) + \ldots.$$

The determinant and trace of the linear terms are -1 and 0. Therefore A is a saddle-point with characteristic roots $+1$ and -1. From the characteristic roots the directions of the separatrices can at once be found. One separatrix diverges from A in the direction of the line $v + 1 = -\frac{1}{2}(\lambda + 5)\,u$, the other separatrix is the line $u = 0$ and converges towards A.

The critical point B is not elementary, since the corresponding linear terms have determinant zero. Nevertheless, the phase portrait in the neighbourhood of B can be obtained by the methods of Keil (1955). If we put

$$w = (2 - \lambda)\,u + v,$$

then

$$\dot{w} = w + 6u^2 - \{(2 - \lambda)\,u - w\}\{(2\lambda + 3)\,u + w\} \equiv w + f(u, w),$$
$$\dot{u} = u\{(2 - \lambda)\,u - w\} \equiv g(u, w).$$

The line $u = 0$ is a path and diverges from B. According to Keil's first theorem no other path leaves the origin in the same direction. The behaviour of the other paths depends on the way in which the ratio $\dot{w}/\dot{u}$ changes sign on crossing the locus $\dot{w} = 0$. In the neighbourhood of the origin the equation $\dot{w} = 0$ has the solution

$$w = \lambda(1-2\lambda)\,u^2+\ldots.$$

Hence when $\dot{w}$ vanishes $\dot{u}$ has the sign of $(2-\lambda)\,u^2$, which is positive. Therefore as w increases and crosses the locus $\dot{w} = 0$ the ratio $\dot{w}/\dot{u}$ changes sign from $-$ to $+$. By Keil's second and third theorems this implies that there is a unique path in the half-plane $u < 0$ which approaches the origin in the direction of the line $v = (\lambda-2)\,u$. Every other path remains in the neighbourhood of the origin for at most a finite interval of time.

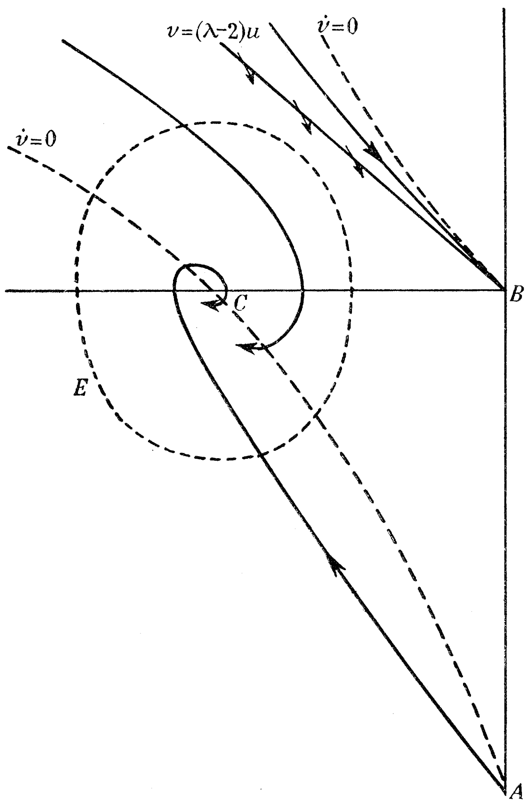

FIGURE 1. Phase portrait for $0 \leqslant \lambda < \frac{1}{2}$.

Consider finally the third critical point C. In the neighbourhood of C

$$\dot{u} = -\rho v+\ldots,$$
$$\dot{v} = (\lambda-2)\,(u-\rho)+\tfrac{1}{6}(7\lambda-8)\,v+\ldots.$$

The determinant and trace of the linear terms are $D = \frac{1}{6}(2-\lambda)^2$ and $T = \frac{1}{6}(7\lambda-8)$. Since D is positive the critical point is stable or unstable according as $T \lessgtr 0$, i.e. according as $\lambda \lessgtr \frac{8}{7}$. Moreover, the critical point is a focus or a node according as $4D \gtrless T^2$. This inequality reduces to

$$25\lambda^2-16\lambda-32 \lessgtr 0.$$

Thus, putting $$\lambda^* = 4(2+3\surd 6)/25 = 1{\cdot}496\ldots,$$

we have a focus for $0 \leqslant \lambda < \lambda^*$ and a node for $\lambda^* < \lambda < 2$.

The critical points having been determined, the next step is to plot the separatrices. By a separatrix we mean a path which tends to a critical point, although neighbouring paths do not tend to this critical point. In order to do this we require some information about the gradient field, in particular about the level curves (isoclines) on which the gradient is zero or infinite.

The equation $\dot{v} = 0$ has the solutions

$$v = -\tfrac{1}{2}(7u+1) \pm \tfrac{1}{2}[25u^2+2(2\lambda+3)\,u+1]^{\frac{1}{2}}.$$

The roots of the quadratic inside the square brackets are

$$u_1, u_2 = -(2\lambda+3)/25 \pm 2[(\lambda+4)\,(\lambda-1)]^{\frac{1}{2}}/25.$$

If $0 \leqslant \lambda < 1$ there are two real values of v for which $\dot{v} = 0$ for every real value of u. If $1 < \lambda < 2$ there are two real values of v, or none, for which $\dot{v} = 0$ according as u lies outside or inside the interval $u_2 \leqslant u \leqslant u_1$. For $\lambda = 1$ the locus $\dot{v} = 0$ consists of the pair of straight lines $v = -u$, $v = -6u-1$.

The slope of the hyperbola $\dot{v} = 0$ at the critical points is easily obtained by differentiation. We have

$$2-\lambda+12u+7v+(1+7u+2v)\,\mathrm{d}v/\mathrm{d}u = 0.$$

It follows that the slope at A is $-(\lambda+5)$, at B is $\lambda-2$, and at C is $6(2-\lambda)/(7\lambda-8)$. If $0 \leqslant \lambda < 1$ the two branches of the hyperbola are strictly decreasing functions of u.

When $v = (\lambda-2)\,u$,

$$\dot{v} = (6u+v)\,(u+v) = (\lambda+4)\,(\lambda-1)\,u^2.$$

Therefore the straight line $v = (\lambda-2)\,u$ lies between the two branches of the hyperbola $\dot{v} = 0$ if $0 \leqslant \lambda < 1$. The field gradient is the same at all points of this line, being equal to

$$(\lambda+4)\,(\lambda-1)/(2-\lambda).$$

This is less than $\lambda-2$ if $0 < \lambda < \frac{1}{2}$ and greater than $\lambda-2$ if $\frac{1}{2} < \lambda < 2$. Hence a path which meets the line $v = (\lambda-2)\,u$ crosses it from above to below if $0 < \lambda < \frac{1}{2}$, and from below to above if $\frac{1}{2} < \lambda < 2$. In particular, the separatrix which converges to B lies always above the line $v = (\lambda-2)\,u$ if $0 < \lambda < \frac{1}{2}$, and always below it if $\frac{1}{2} < \lambda < 2$. For $\lambda = 0$ and $\lambda = \frac{1}{2}$ the line $v = (\lambda-2)\,u$ is itself the separatrix.

When v vanishes $\dot{v} = 6u(u-\rho)$. Therefore *a path which intersects the u-axis crosses it from above to below or below to above according as the abscissa of the point of intersection is greater or less than* ρ.

Let

$$U = \tfrac{1}{2}v^2+3(u-\rho)^2. \tag{21}$$

It is easily verified that

$$\dot{U} = v^2(v+7u+1).$$

Thus U is a decreasing function of τ below the line $v+7u+1 = 0$. The minimum value of U on this line is $(7\lambda-8)^2/660$ and is attained for $u = (\lambda-9)/55$. It follows that if $(\lambda-9)/55 > \rho$, that is to say if $\lambda < \frac{8}{7}$, then any path which passes through a point inside the ellipse

$$\tfrac{1}{2}v^2+3(u-\rho)^2 = (7\lambda-8)^2/660 \tag{E}$$

remains inside the ellipse ever after. Moreover, U decreases and tends to a non-negative limit $\overline{U}$ as $\tau \to \infty$. We will show that $\overline{U} = 0$.

It is sufficient to show that $v \to 0$. For then u also has a finite limit and this limit must be ρ, because C is the only critical point inside the ellipse. If v does *not* tend to zero there exists a sequence $\tau_n \to \infty$ such that $u(\tau_n) \to a$, $v(\tau_n) \to b \neq 0$. We can find positive numbers δ and η such that $\dot{U} \leqslant -\eta$ throughout the neighbourhood $|u-a| \leqslant \delta$, $|v-b| \leqslant \delta$ of the point (a, b). Let M be an upper bound for $|\dot{u}|$ and $|\dot{v}|$ in this neighbourhood and choose a value τ_n so large that

$$|u(\tau_n)-a| < \tfrac{1}{2}\delta, \quad |v(\tau_n)-b| < \tfrac{1}{2}\delta, \quad U(\tau_n)-\overline{U} < \delta\eta/2M.$$

The point $u(\tau)$, $v(\tau)$ remains in the neighbourhood for $\tau_n \leqslant \tau \leqslant \tau_n+\delta/2M$, by the definition of M. Therefore

$$U(\tau_n)-\overline{U} \geqslant U(\tau_n)-U(\tau_n+\delta/2M) \geqslant \delta\eta/2M.$$

This is the required contradiction. Thus we have proved that *if* $\lambda < \frac{8}{7}$ *every path which passes through a point in the interior of the ellipse* (E) *converges to the critical point* C *as* $\tau \to \infty$.

Consider now the separatrix which leaves the critical point A. Its slope at A is $-\frac{1}{2}(\lambda+5)$, whereas the slope of the hyperbola $\dot{v} = 0$ at this point is $-(\lambda+5)$. Therefore the separatrix passes into the domain where $\dot{v} > 0$. Moreover, it remains in this domain for as long as it is in the quadrant $u < 0$, $v < 0$, since $\dot{u}$ is negative in this quadrant. Thus the representative point moves upwards and to the left. We will show that the separatrix eventually intersects the u-axis. It is sufficient for this purpose to show that if $0 < m < \frac{1}{2}(\lambda+5)$ then every path which meets the straight line $v = -1-mu$ in the quadrant $u < 0$, $v < 0$ crosses it from below to above. The field gradient at any point (u, v) on this line is

$$\frac{2-\lambda+6u}{1+mu}-7+m.$$

Since the linear fractional function has a positive determinant the gradient is a continuous increasing function of u for $m^{-1} < u \leqslant 0$. For $u = 0$ the gradient is

$$-(\lambda+5)+m < -\tfrac{1}{2}(\lambda+5) < -m.$$

Therefore the gradient is everywhere less than $-m$, which proves the assertion. Since m may be arbitrarily close to $\frac{1}{2}(\lambda+5)$ the separatrix intersects the u-axis at a point whose abscissa is not less than $-2/(\lambda+5)$.

We can now show that *the separatrix which diverges from* A *converges to* C *as* $\tau \to \infty$ *if* $0 \leqslant \lambda \leqslant \frac{1}{2}$. For it intersects the u-axis at a point whose abscissa lies between $-2/(\lambda+5)$ and ρ. It then moves upwards and to the right until it meets the lower branch of the hyperbola $\dot{v} = 0$. If $\lambda \leqslant \frac{1}{2}$ it cannot cross the line $v = (\lambda-2)\,u$ and consequently it cannot meet the upper branch of the hyperbola. It therefore drops steadily downwards. Moreover, it cannot intersect the u-axis until u is greater than ρ. Thus the ordinate of the separatrix when $u = \rho$ is positive and less than the ordinate of the lower branch of the hyperbola when $u = -2/(\lambda+5)$. A simple calculation shows that the latter ordinate does not exceed $1/(\lambda+5)$. Hence the ordinate of the separatrix when $u = \rho$ is definitely less than $\frac{1}{5}$ and the separatrix lies inside the elipse (E) at this point. Therefore, by what has already been proved, it converges to C as $\tau \to \infty$.

Still supposing $0 \leqslant \lambda \leqslant \frac{1}{2}$, we will show further that *every path which passes through a point of the domain* $u < 0$, $v < (\lambda-2)\,u$ *converges to* C *as* $\tau \to \infty$. In the first place no path ever leaves this domain. The argument used at the end of the previous section shows also that u and v are bounded as $\tau \to \infty$ on any path in this domain. If the path did not tend to the critical point C its limit-set would be a closed path containing C in its interior, by the theorem of

Poincaré–Bendixson. But the path would then intersect the separatrix from A, which is impossible. Similar reasoning shows that *the only path which remains in a bounded portion of the domain* $u < 0$, $v < (\lambda - 2)\,u$ *as* $\tau \to -\infty$ *is the path consisting of the single point* C.

It is more difficult to determine what happens when $\lambda > \frac{1}{2}$. The separatrix which converges to B now lies below the line $v = (\lambda - 2)\,u$ and may intersect the u-axis. By the theory

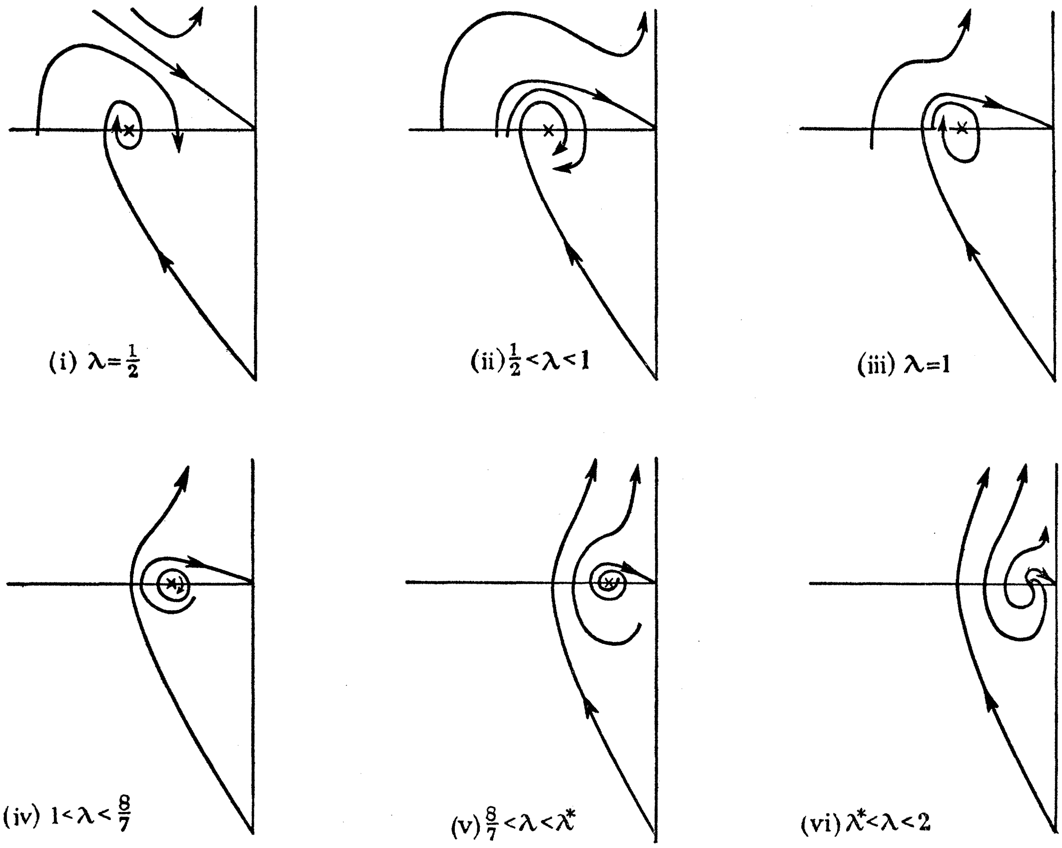

FIGURE 2. Phase portraits for $\frac{1}{2} \leqslant \lambda < 2$.

of structural stability the phase portrait remains qualitatively the same as λ increases until the separatrix which diverges from A fuses with the separatrix which converges to B.† This fusion actually takes place. In fact the parabola

$$u + v + (2u + v)^2 = 0$$

passes through the critical points A and B and is a path for $\lambda = 1$ (or rather, the part of it which lies in the half-plane $u < 0$ is a path). I have not been able to prove that there is no other value of λ for which fusion takes place, although this is almost certainly the case. The difficulty arises from the fact that the intersections of the two separatrices with the u-axis *both* move to the right as λ increases. In all other examples known to me of this phenomenon the intersections of the two separatrices with a fixed line move in opposite directions, from which it is obvious that they coincide at most once. If it could be proved that fusion does not occur before $\lambda = 1$ then the classification of solutions of the boundary-layer equation announced in § 1 would be valid for all values of $\lambda \leqslant 1$.

† The critical point B is not elementary, but the phase portrait in the neighbourhood of B does not alter during the variations which we consider.

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By transforming to polar co-ordinates—the procedure is described in Andronov & Chaikin (1949)—it may be shown that for values of λ slightly less than $\frac{8}{7}$ there is an unstable limit-cycle containing C in its interior which contracts to C as $\lambda \to \frac{8}{7}$. The limit-cycle disappears on passing the value $\lambda = \frac{8}{7}$ at which the critical point becomes unstable. The complete sequence of phase portraits thus appears to be as shown in figure 2.

9. The equation of Blasius ($\lambda = 0$)

The non-negative values $\lambda = 0$ and $\lambda = \frac{1}{2}$ have hitherto been excluded from discussion. They are of particular importance for the applications, however, and must now be considered.

For $\lambda = 0$ the equation (1) takes the simple form

$$y''' + yy'' = 0. \tag{22}$$

If $y = \phi(x)$ is a solution of (22) then $y = c\phi(cx)$ is also a solution, for any constant c. By theorem 5 the second derivative of any solution vanishes identically or not at all.

Suppose first that y'' *is always positive*. If y were defined only in a finite interval $0 \leqslant x < x^*$ then y' and y would tend to $+\infty$ as $x \to x^*$, by theorem 4. Therefore, by the differential equation (22), y''' is negative near x^*. Thus y'' is a positive decreasing function, and consequently bounded, near x^*. But this contradicts theorem 4. Therefore y is defined for all $x \geqslant 0$.

As $x \to \infty$ y' tends to a limit k $(-\infty < k \leqslant \infty)$. We will show that k is necessarily finite and non-negative. If $k > 0$ then $y > 1$ for all $x \geqslant x_0$, say. Therefore $y''' < -y''$ and hence

$$y'' < y''_0 \, e^{-(x-x_0)} \quad \text{for} \quad x \geqslant x_0.$$

It follows on integration that y' is bounded for $x \to \infty$, which proves that k is finite. For all large x $y''' = -yy''$ is of constant sign. Therefore y'' has a definite limit as $x \to \infty$. This limit must be zero, since y' has a finite limit. Therefore y''' is negative and y is positive for all large x. Since $y \sim kx$ this implies that $k \geqslant 0$. Moreover, if $k = 0$ then y is a decreasing function and tends to a finite non-negative limit as $x \to \infty$. Thus there are three possibilities as $x \to \infty$:

$$y \to 0, \quad y' \to 0, \quad y'' \to 0,$$

$$y \to \alpha, \quad y' \to 0, \quad y'' \to 0 \quad (\alpha > 0),$$

$$y \sim \beta x, \quad y' \to \beta, \quad y'' \to 0 \quad (\beta > 0).$$

The solution is of the last type if initially $y' \geqslant 0$, or $y' < 0$ and $y \leqslant 0$. In particular, if $y = y' = 0$ initially, then $y = \phi(x)$ is defined for all $x \geqslant 0$ and y' has a positive limit β as $x \to \infty$. By replacing $\phi(x)$ by $c\phi(cx)$, where $c = \beta^{-\frac{1}{2}}$, we can arrange that this limit is 1. Therefore *the differential equation* (22) *has a solution which satisfies the boundary conditions*

$$y = y' = 0 \quad \text{at} \quad x = 0, \qquad y' \to 1 \quad \text{as} \quad x \to \infty.$$

The uniqueness proof given in § 2 is still valid for $\lambda = 0$.

Suppose next that y'' *is always negative*. Then y' is a decreasing function. If y and y' are both negative initially then they are negative for all $x > 0$ for which the solution is defined. But $\bar{y}(x) = 3/(x - x_0)$ is a solution of (22) for any constant x_0. If we choose $x_0 > 0$ so large that

$$y(0) \leqslant -3x_0^{-1}, \quad y'(0) \leqslant -3x_0^{-2}, \quad y''(0) \leqslant -6x_0^{-3},$$

then by Kamke's theorem $y(x) \leqslant \bar{y}(x)$ for all $x > 0$ for which y is defined. It follows that y is defined only in a finite interval $0 \leqslant x < x^*$.

If y is defined only in a finite interval $0 \leqslant x < x^*$ then $y' \to -\infty$ and $y \to -\infty$ as $x \to x^*$, by theorem 4. Therefore y''' is negative near x^* and $y'' \to -\infty$ as $x \to x^*$, by theorem 4 again.

If y is defined for all $x \geqslant 0$ then y' tends to a limit k as $x \to \infty$ $(-\infty \leqslant k < \infty)$. We cannot have $k < 0$. For in this case y' and y would ultimately both be negative and the solution would be defined only in a finite interval, by what has already been proved. Therefore y' tends to a finite non-negative limit. Since y'' is ultimately monotonic it must tend to zero. Therefore y''' and y are ultimately positive.

We show finally that y tends to a finite limit when $y' \to 0$. In fact if $y > 1$ for $x \geqslant x_0$ then $y''' > -y''$, and hence

$$y'' > y_0'' \, \mathrm{e}^{-(x-x_0)}.$$

Integrating between the limits x and ∞ we get

$$0 < y'(x) < -y_0'' \, \mathrm{e}^{-(x-x_0)}.$$

Hence y is bounded for $x \to \infty$. Since it is an increasing function the limit to which it tends is positive.

Thus there are altogether three possibilities:

$$\begin{array}{llllll} y \to -\infty, & y' \to -\infty, & y'' \to -\infty & \text{as} & x \to x^*, & \\ y \to \alpha, & y' \to 0, & y'' \to 0 & \text{as} & x \to \infty & (\alpha > 0), \\ y \sim \beta x, & y' \to \beta, & y'' \to 0 & \text{as} & x \to \infty & (\beta > 0). \end{array}$$

The solution is of the first type if $y' \leqslant 0$ initially.

We will now determine the asymptotic forms of the various types of solution. This has already been done in § 4 for the solutions for which $y' \to 1$, and the asymptotic form of the solutions for which $y' \to \beta$ follows merely by a change of scale. The form of the solutions for which $y \to \alpha$ can be found by the same sort of argument. In fact if $y \to \alpha$ then

$$y'' = \mathcal{O}(\mathrm{e}^{-\alpha x}), \quad y' = \mathcal{O}(\mathrm{e}^{-\alpha x}), \quad y - \alpha = \mathcal{O}(\mathrm{e}^{-\alpha x})$$

for $x \to \infty$. Writing the equation (22) in the form

$$y''' + \alpha y'' = (\alpha - y)\, y'' \equiv f(x)$$

we obtain, by the variation of constants formula,

$$y'' = c\, \mathrm{e}^{-\alpha x} - \int_x^\infty \mathrm{e}^{\alpha(\xi - x)} f(\xi)\, \mathrm{d}\xi,$$

where $f(x) = \mathcal{O}(\mathrm{e}^{-2\alpha x})$. Hence

$$y'' = c\, \mathrm{e}^{-\alpha x} + \mathcal{O}(\mathrm{e}^{-2\alpha x}),$$

and the asymptotic forms of y' and y follow by integration. More accurate approximations can be found by successive substitution. Thus $f(x) \sim -(c/\alpha)^2\, \mathrm{e}^{-2\alpha x}$ and hence

$$y'' \approx c\, \mathrm{e}^{-\alpha x} + c^2 \alpha^{-3}\, \mathrm{e}^{-2\alpha x}.$$

It will now be shown that *for any given constants α and c $(\alpha > 0)$ the equation* (22) *has one and only one solution defined for all sufficiently large x such that*

$$y \to \alpha, \quad y'' \sim c\, \mathrm{e}^{-\alpha x} \quad \textit{as} \quad x \to \infty.$$

This is equivalent to showing that the integro-differential equation

$$Y''(x) = c\,e^{-\alpha x} + \int_x^\infty e^{\alpha(\xi-x)}\,Y(\xi)\,Y''(\xi)\,d\xi$$

has a unique solution for which

$$Y'' = O(e^{-\alpha x}), \quad Y \to 0 \quad \text{as} \quad x \to \infty.$$

Let $Y(x)$ be any twice continuously differentiable function such that

$$|Y''(x)| \leqslant (1+c^2)\,e^{-\alpha x} \quad \text{for} \quad x \geqslant 0, \qquad Y(x) \to 0 \quad \text{as} \quad x \to \infty.$$

Then there is a unique twice continuously differentiable function $Z = \mathscr{T}Y$ tending to zero as $x \to \infty$ for which

$$Z''(x) = c\,e^{-\alpha x} + \int_x^\infty e^{\alpha(\xi-x)}\,Y(\xi)\,Y''(\xi)\,d\xi.$$

Moreover $$|Z''(x) - c\,e^{-\alpha x}| \leqslant (1+c^2)^2\,\alpha^{-3}\,e^{-2\alpha x},$$

since $$|Y(x)| \leqslant (1+c^2)\,\alpha^{-2}\,e^{-\alpha x}.$$

If we choose $x_0 > 0$ so large that

$$(1+c^2)^2\,\alpha^{-3}\,e^{-\alpha x_0} \leqslant \tfrac{1}{4}$$

we will certainly have

$$|Z''(x)| < (1+c^2)\,e^{-\alpha x} \quad \text{for} \quad x \geqslant x_0.$$

Now set $$Y_0 \equiv 0, \quad Y_{n+1} = \mathscr{T}Y_n \quad (n = 0, 1, 2, \ldots).$$

By the definition of the transformation $\mathscr{T}$

$$Y''_{n+1} - Y''_n = e^{-\alpha x}\int_x^\infty e^{\alpha\xi}\,(Y_n Y''_n - Y_{n-1} Y''_{n-1})\,d\xi.$$

Suppose $$|Y''_n(x) - Y''_{n-1}(x)| \leqslant A_n\,e^{-\alpha x} \quad \text{for} \quad x \geqslant x_0.$$

This is certainly true for $n = 1$, with $A_1 = 1+c^2$. Writing the difference $Y_n Y''_n - Y_{n-1} Y''_{n-1}$ in the form

$$(Y_n - Y_{n-1})\,Y''_n + Y_{n-1}(Y''_n - Y''_{n-1})$$

it follows that

$$\begin{aligned}|Y''_{n+1}(x) - Y''_n(x)| &\leqslant 2A_n(1+c^2)\,\alpha^{-3}\,e^{-2\alpha x}\\ &\leqslant \tfrac{1}{2}A_n\,e^{-\alpha x} \quad \text{for} \quad x \geqslant x_0.\end{aligned}$$

Therefore the series $\Sigma\{Y''_n - Y''_{n-1}\}$ converges at least as fast as a geometric series with common ratio $\frac{1}{2}$. So also does the series obtained by integrating twice term by term. Thus $Y_n \to Y$ and $\mathscr{T}Y_n \to \mathscr{T}Y$, from which we conclude that $Y = \mathscr{T}Y$. That is to say, the limit function is a solution of the integro-differential equation.

It remains to prove that the solution is unique. Let Z be any solution. Then

$$Z - Y_{n+1} = e^{-\alpha x}\int_x^\infty e^{\alpha\xi}(ZZ'' - Y_n Y''_n)\,d\xi.$$

By hypothesis there exists a constant B_0 such that

$$|Z''(x)| \leqslant B_0\,e^{-\alpha x} \quad \text{for} \quad x \geqslant x_0.$$

Suppose $$|Z''(x) - Y''_n(x)| \leqslant B_n\,e^{-\alpha x} \quad \text{for} \quad x \geqslant x_0.$$

Then it follows in the same way as before that

$$\begin{aligned}|Z''(x) - Y''_{n+1}(x)| &\leqslant B_n(1+c^2+B_0)\,\alpha^{-3}\,e^{-2\alpha x}\\ &\leqslant \tfrac{1}{2}B_n\,e^{-\alpha x} \quad \text{for} \quad x \geqslant x_1,\end{aligned}$$

where x_1 is independent of n. Hence $Y''_n \to Z''$ and $Y_n \to Z$. Since a sequence cannot converge to two different limits we must have $Y = Z$.

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We turn now to the solutions for which y, y' and $y'' \to -\infty$ as $x \to x^*$. Since $y''' \to -\infty$ also, the corresponding paths ultimately lie in the domain [23]. The differential equation (22) can be transformed into the system studied in §8 with $\lambda = 0$. According to the results established there we have $y' \sim -\frac{1}{3}y^2$ for $y \to -\infty$ on every path in [23]. It follows that

$$y \sim -3/(x^*-x) \quad \text{for} \quad x \to x^*-0.$$

Finally, we consider the solutions for which y, y' and y'' tend to zero as $x \to \infty$. We again make the change of variables $z = y^2u$, $y = e^t$. Since $y'' > 0$, $y' < 0$ and $y > 0$ we have

$$u < 0, \quad \dot{u} < -2u \quad \text{and} \quad t \to -\infty.$$

We will show that u and $\dot{u}$ remain bounded as $t \to -\infty$. The function $y''+yy'$ increases with x, since

$$\frac{d}{dx}(y''+yy') = y'^2,$$

and tends to zero as $x \to \infty$. It follows that

$$dz/dy = y''/y' > -y$$

and hence, by integration,

$$z > -\tfrac{1}{2}y^2.$$

Therefore $u > -\frac{1}{2}$ and $\dot{u} > -1$. But, by §8, the only path which remains in a bounded portion of the domain $u < 0$, $\dot{u} < -2u$ as $t \to -\infty$ is the 'point' path $u \equiv -\frac{1}{3}$, $\dot{u} \equiv 0$. It follows that the solutions for which y tends to zero all have the form

$$y = 3/(x-x_0).$$

To sum up: *apart from the exact solutions*

$$y = \alpha+\beta x, \quad y = 3/(x-x_0)$$

there are three classes of solution—a 3-parameter manifold of solutions for which $y \sim \beta x$ as $x \to \infty$ $(\beta > 0)$, a 3-parameter manifold of solutions for which $y \sim -3/(x^*-x)$ as $x \to x^*-0$, and a 2-parameter manifold of solutions for which $y \to \alpha$ as $x \to \infty$ $(\alpha > 0)$.

10. The equation of Homann ($\lambda = \frac{1}{2}$)

The special case $\lambda = \frac{1}{2}$ is of particular interest because it also describes the three-dimensional flow of a viscous fluid against a plane wall (Homann 1936). It is not, like $\lambda = 0$, a genuinely exceptional case, but is exceptional only as regards the chosen mode of subdividing the phase space.

Every solution of the equation

$$y'''+yy''+\tfrac{1}{2}(1-y'^2) = 0 \tag{23}$$

also satisfies the equation

$$y^{\text{iv}}+yy''' = 0, \tag{24}$$

obtained from it by differentiation. But $y = \alpha+\beta x+\frac{1}{2}\gamma x^2$ is a solution of the latter equation for any values of the constants α, β, γ. It follows that the third derivative of any solution of (24), and *a fortiori* of any solution of (23), vanishes identically or not at all. Therefore every path terminates in one of the 24 domains, except the paths for which $y_3' \equiv 0$.

The same laws of passage still hold for $\lambda = \frac{1}{2}$. In other words, a transition between two domains is permissible for $\lambda = \frac{1}{2}$ if it is permissible for both $\lambda < \frac{1}{2}$ and $\lambda > \frac{1}{2}$. It is easily seen that the only possible terminal domains are again

[1], [2], [3], [4], [22], [23],

even though y_3 may now have a finite non-zero limit. The domains [3] and [4] are already known to be terminal. The domains [1], [2], and [23] are terminal by the laws of passage. In fact, no path ever leaves one of these domains. However, it will now be proved that [22] is a transitional domain. On any path in [22] y_3' is an increasing function, since

$$y_3'' = -y_1 y_3' > 0.$$

If the path remained in [22] for all $x \geqslant 0$ then y_3' would tend to $+\infty$ as $x \to \infty$, by theorem 2. But this implies that $y_3 \to +\infty$, whereas $y_3 < 0$ in [22].

It follows from the laws of passage that every path for which $y'' \geqslant 0$ and $y''' > 0$ initially terminates in [1], and every path for which $y'' \leqslant 0$ and $y''' < 0$ initially terminates in [23].

The asymptotic form of the solutions which terminate in [3] or [4] is already known from § 4. The argument used in § 7 (i) and (iii) to obtain the form of the solutions which terminate in [1] and [2] is still valid for $\lambda = \frac{1}{2}$. Hence for these solutions

$$z \approx ay^{\frac{1}{2}} + \frac{1}{2ay^{\frac{1}{2}}} \quad \text{for} \quad y \to \infty \quad (a > 0),$$

and $y \sim bx^2$ for $x \to \infty$ $(b > 0)$. Again, the argument of § 7 (iv) shows that for all solutions which terminate in [23]

$$z \sim -\tfrac{1}{4}y^2 \quad \text{for} \quad y \to -\infty.$$

Finally, if $b < 0$ the solutions $y = (x^2 - b^2)/2b$ for which $y''' \equiv 0$ have the asymptotic form

$$z \sim a\,|y|^{\frac{1}{2}} \quad \text{for} \quad y \to -\infty \quad (a < 0).$$

In short, the classification of solutions stated in § 1, which has already been established for $0 < \lambda < \frac{1}{2}$, continues to hold for $\lambda = \frac{1}{2}$.

11. Final remarks

It is sometimes argued that because the equation (1) is of the third order one can reasonably expect that it has a unique solution satisfying the three boundary conditions (2). However, this 'counting' principle, although it may have some value for linear equations, is really worthless for non-linear equations. The solubility of a boundary-value problem can only be understood in the light of a survey of the solutions as a whole. On these grounds alone the present investigation seems justified.

There are two respects in which the investigation remains incomplete. The existence of oscillatory solutions, and their asymptotic form if they do exist, has been left undecided for $\lambda > 1$. Also, for $\frac{1}{2} < \lambda < 2$ we have not found the asymptotic form of all solutions for which y, y' and y'' tend to $-\infty$. The two matters are not unrelated, because these solutions may turn into oscillatory solutions for a certain value of λ.

The solubility of the boundary-value problem for *negative* values of λ was examined by Hartree and also, more recently, by Stewartson (1954). Without doubt one can discuss all solutions of the boundary-layer equation for $\lambda < 0$ in the way we have done here for $\lambda \geqslant 0$. But it did not seem worthwhile to do this, first because the physical significance of negative values of λ is not completely clear, and secondly, because the paper is already long enough.

Theorem 1, and more generally the equivalence between the theory of functions of a complex variable and the theory of perfect fluids, suggest that it may be profitable to study the boundary-layer equation in the complex domain.

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The only paper on the boundary-layer equation, other than those listed above, which I had seen before or during the writing of the present paper was a note by S. Furuya, *Comment. Math. Univ. St Paul*, **1** (1953), 81–83, on the uniqueness of the solution of the boundary-value problem. After the manuscript was completed I made a systematic search of the literature, which revealed the following papers having contact with mine:

R. Iglisch, *Z. angew. Math. Mech.* **33** (1953), 143–7 and **34** (1954), 441–3.

Iglisch's work is closely related to §2 of the present paper. As he supposes the initial value of y' to be zero, but allows the initial value of y to be any real number, his final result neither contains nor is contained in mine.

A. D. Myschkis & G. V. Gil, *Dokl. Akad. Nauk SSSR*, **112** (1957), 599–602.

This contains some asymptotic formulae for the solution of the boundary-value problem. The results are not as exact as the 'less precise' formula (8*b*) derived in §4 and the method of derivation is not so direct.

C. W. Jones, *Proc. Roy. Soc.* A, **217** (1953), 327–343.
B. Punnis, *Archiv der Math.* **7** (1956), 165–171.

It is pointed out in Weyl's paper, where the observation is attributed to von Neumann, that the equation of Blasius can be reduced to a first-order equation and hence is capable of treatment by the methods of Poincaré. Jones gives a canonical form for this first-order equation but does not investigate it further. Punnis uses this equation to show that the solution of the boundary-value problem has the form $y \sim 3/(x-x^*)$ as $x \to x^*$ for some $x^* < 0$. (His reasons for asserting that the path in phase-space necessarily tends to a critical point are not clear to me.) It is remarkable that even in the classical case of Blasius there has been no attempt to study *all* solutions of the equation.

Note added in proof, 14 *July* 1960

The upper bound for $\gamma = y''(0)$ obtained in § 3 was derived on the assumption that $\alpha = 0$ and $\lambda \geqslant \frac{1}{2}$. The following much simpler proof shows that the restriction on λ may be removed and at the same time improves the bound itself.

It is shown in § 3 that $\frac{1}{2}y'^2 - yy''$ is an increasing function of x. Therefore, if $\alpha = 0$,

$$\tfrac{1}{2}y'^2 - yy'' > \tfrac{1}{2}\beta^2 \quad \text{for} \quad x > 0.$$

It follows that $z(x) = y'(x)$ satisfies the differential inequality

$$z'' > (\lambda - \tfrac{1}{2})\, z^2 + \tfrac{1}{2}\beta^2 - \lambda,$$

and $v(x) = z'(x)$ satisfies the differential inequality

$$v\,\mathrm{d}v/\mathrm{d}z > (\lambda - \tfrac{1}{2})\, z^2 + \tfrac{1}{2}\beta^2 - \lambda.$$

Integrating with respect to z between the limits β and 1 we get

$$-\tfrac{1}{2}\gamma^2 > \tfrac{1}{3}(\lambda - \tfrac{1}{2})\,(1-\beta^3) + (\tfrac{1}{2}\beta^2 - \lambda)\,(1-\beta),$$

since $v = \gamma$ when $z = \beta$ and $v \to 0$ as $z \to 1$. After a little rearrangement this gives the following inequality:

If $\alpha = 0$ the initial value γ of y'' which corresponds to the solution of the boundary-value problem satisfies the inequality

$$\gamma^2 < \tfrac{1}{3}(1-\beta)^2\{2\lambda(2+\beta) + 1 + 2\beta\}.$$

The proof of the lower bound for γ may be simplified in the same way.